ON A LOCAL-GLOBAL PRINCIPLE FOR INTEGRAL QUADRATIC FORMS

Jonathan Hanke

Abstract. In this paper we investigate the representation behavior of a positive definite ternary quadratic form Q within an exceptional/anti-exceptional square class $t\mathbb{Z}^2$. We show that a local-global principle based on the spinor genus holds for all but finitely many numbers outside of finitely many infinite sequences mq^2 where $q \nmid N$ runs over primes with $\psi(q) = -1$ (with the assumption that m has bounded divisibility at the anisotropic primes of Q). In the process, we describe the structure of the weight 2 cusp form $g(z)$ associated to Q along $t\mathbb{Z}^2$, showing that this interesting portion of it arises as a difference/sum of a form and its twist by ψ. We conclude that one can completely understand which numbers are represented by any quadratic form in 3 or more variables.

§0 Introduction and Notation

Given a diophantine equation with integer coefficients, a central problem of number theory is determining whether integer solutions exist. Here we will attempt to address the question for the equation

$$Q(\vec{x}) = m \tag{0.1}$$

where Q is an integral quadratic form in ≥ 3 variables and m is an integer. For convenience we let $r_Q(m)$ denote the number of solutions and say that m **is represented by** Q when $r_Q(m) > 0$.

One approach to this problem is to first ask if there are solutions mod p^α for all primes p and all $\alpha > 0$, as well as over $\mathbb{R}$. If there are, we call them **local solutions** and say that m **is locally represented by** Q. The existence of local solutions is a necessary for m to be represented by Q, but it is not in general sufficient. For those m where it is sufficient, we say that a **local-global principle** exists.

If we are interested in rational solutions to (0.1) rather than integral ones, then the existence of a local-global principle goes back to Hasse [Has] and is one of the major achievements of the thoery of quadratic forms (however at every prime p we ask for solutions over the p-adic numbers $\mathbb{Q}_p$). For integral solutions, the situation is complicated by the fact that there may be finitely many (non-isomprphic) forms which locally look like Q. We call the set of such forms the **genus of** Q and

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denote it by Gen(Q). Notice that when a local-global principle exists at m, it makes a uniform statement about all forms in the genus, saying $r_{Q'}(m) > 0$ for all $Q' \in \mathrm{Gen}(Q)$.

It was Siegel who first quantitatively described the relationship between the number of local and global representations, expressing a certain weighted average over $r_{Q'}(m)$ all $Q' \in \mathrm{Gen}(Q)$ as an infinite product of local representation densities $\beta_v(m)$ over all places v of $\mathbb{Q}$. Siegel's Theorem gives a completely satisfactory local-global principle so long as we are content to understand an average number or representations over all forms in the genus. But what can be said for individual forms?

If we are fortunate enough that $\mathrm{Gen}(Q) = \{Q\}$, then a local-global principle holds for all m. However in a pioneering paper [Wa], Watson he gave several examples of ternary quadratic forms Q and infinite families of numbers m on which a local-global principle fails.

Upon further investigation [???,???], we see that the problems pointed to by Watson are related to a more refined local equivalence (spinor equivalence) which breaks the genus into finitely many **spinor genera**. (There is also an analagous formula to Siegel's Theorem which holds over each spinor genus in the genus.) By changing our the local criterion to be representability by (some form Q' in) the spinor genus Spn(Q) instead of the genus Gen(Q), we avoid Watson's problems and may again ask about the validity of a local-global principle.

In attempting to answer this question, the theory breaks naturally into two parts depending on the behavior of Q over $\mathbb{R}$. If Q represents all of $\mathbb{R}$, then it is called **indefinite** and it is known [Si] that there is exactly one form in each spinor genus. Thus for these forms one has a local-global principle (and even an explicit description of the number of representations in a certian sense). If Q represents only half of $\mathbb{R}$ we say it is **definite**, in which case there may be many forms in each spinor genus. In this case, the formula of Siegel yields no information about the behavior of an individual form and there may be numbers which are locally represented by Q, but not represented by Q globally.

To pursue a local-global principle further, clearly a different perspective is required. At this point, one can try to use analytic methods to estimate how the number of representations $r_Q(m)$ grows with m. When the error in this estimate becomes small as m grows, we have that a local-global principle holds for m sufficiently large. To understand how this plays out, we describe some of the main features of this approach, and why they lead to certain subtleties for forms in 3 or 4 variables.

One begins by studying the theta function

$$\Theta_Q(z) = \sum_{m \in \mathbb{Z}_{\geq 0}} r_Q(m)\, e^{2\pi i m z} \tag{0.2}$$

associated to a positive definite form Q in n variables. This is a generating function for the representation behavior of Q on integers, as well as being a modular form of weight $n/2$ for some congruence subgroup $\Gamma_0(N)$. If n is odd, then this is a form of half-integral weight in the sense on Shimura [Sh1].

One then considers a decomposition $\Theta_Q(z) = E(z) + f(z)$ where E is an Eisenstein series and f is a cusp form. In this language, Siegel's theorem states that the

Eisenstein series E can be written as a weighted average of theta functions over the genus, and that its Fourier coefficients $a_E(m)$ can be expressed as certain infinite local products. By computing these local factors, one can obtain effective statements about the growth of $a_E(m)$ which is our main term, and by Deligne's bound on the growth of the cusp coefficients, we obtain our error term. This simple picture completely describes the situation for $n \geq 5$, but it becomes more complicated for smaller n.

For $n = 3$ or 4, we see the possible emergence of finitely many **anisotropic** primes at which the local factors contributing to the growth of $a_E(m)$ are constant as m grows at p (meaning m grows by introducing additional factors of p). Growth at these primes do not contribute to the asymptotic for $a_E(m)$, and so require restrictions on their divisibility when we ask m to be sufficently large. These primes all divide the determinant d of Q, and so are easily identified.

For $n = 3$, we have several additional complications. The first is that the constant in the asymptotic describing the growth of $a_E(m)$ involves the value $L(1, \chi_t)$ where t is the square free part of m. Since the lower bound $|L(1, \chi_t)| \geq C_\varepsilon t^{\frac{1}{2} - \varepsilon}$ is ineffective, so is the estimate for our main term. The resolution of this problem is equivalent to knowing there are no Siegel zeros for Dirichlet L-functions and would follow from the Riemann hypothesis.

The second problem involves the bound for the error term, i.e., the bound for Fourier coefficients of half-integral weight cusp form. The growth of Fourier coefficients within a square class can be described in terms of the bound for even weight forms via the Shimura lift, leaving the case of bounding the square free coefficients. Here the convexity bound $O(t^{1/2})$ just barely fails to meet our needs. However in a major achievement, this exponent was improved by Iwaniec [I1] in the case of weight $\geq 5/2$ and then extended (ineffectively) in [Du-SP] to weight $3/2$, however by slightly different methods one can prove an effective bound of this kind.

The third problem is the presence of weight $3/2$ cusp forms whose Shimura lift is not cuspidal. The relationship of these forms to Q was described by Schulze-Pillot [SP1] and its component in Θ_Q depends only on the **half genus** of Q. **(Say something about the spinor rep'n behavior depending only on the genus for $n > 3$ and half-genus when $n = 3$).** These exceptional cusp forms are well understood as modular forms and have non-zero Fourier coefficients on only finitely many square classes. They also have the same order of growth as the main term $a_E(m)$, and their presence is responsible for the infinite exceptional sequences pointed out by Watson. On such square classes our (combined) main term may be exactly zero infinitely often, which is problematic for our asymptotic approach. Further, this would indicate the possibly unpleasant siutation that the representation behavior would depend on the underlying Fourier coefficients of the cusp forms with cuspidal Shimura lift. Since the behvaior of the coefficients of cusp forms are not generally understood, this might lead us to believe that the question of the validity of a local-global principle in such cases may well be beyond our reach (see [SP2]).

In this paper, we show that this concern is unwarranted and determine the exact nature of the local-global principle for ternary forms within these exceptional square classes and in their anti-exceptional counterparts (in the other half genus). We do this by first describing the cuspidal error contribution (§1) in terms of the Shimura lift, and describing an asymptotic formula for our main term along the

exceptional/anti-exceptional $t\mathbb{Z}^2$ (§2). Then we treat the exceptional and anti-exceptional cases separately (§4 and §5), appealing to a description of weight 2 cusp forms whose p^{th} Fourier coefficients vanish according to some quadratic character ψ (§3) to see that the cuspidal contribution to $\Theta_Q(z)$ is well-behaved. This follows from analyzing the associated Galois representations. We conclude with an example (§6) where we use our results to determine which numbers are represented by Q within the exceptional/anti-exceptional square classes, and some closing comments (§7) related to growth at the anisotropic primes and the structure of $g(z)$.

Notation

We let $\mathbb{Z}, \mathbb{Q}\, \mathbb{R}$, and $\mathbb{C}$ denote the integers, rational, real, and complex numbers respectively. We also let $\mathbf{e}(z) = e^{2\pi i z}$, $\tau(m) = \sum_{d|m, d>0} 1$, $\mu(m)$ denote the Möebius function, $\left(\frac{a}{b}\right)$ denote the Jacobi symbol, and $gcd(a, b)$ be the greatest common divisor of a and b. For a square matrix α, we let $Tr(\alpha)$ and $Det(\alpha)$ denote its trace and determinant. We write $m >> 1$ to represent a condition that is true when m is sufficiently large.

For any number field L we let $G_L = Gal(\overline{L}/L) \subseteq G_{\mathbb{Q}}$. If λ is a prime number/ideal in the integers O_L of L, we let $\mathbb{F}_\lambda$ denote the residue field $O_L/\lambda O_L$, and let $Frob_\lambda \in G_L$ denote the Frobenius element associated to λ.

For a Dirichlet chracter Φ, we denote the associated Dirichlet L-function by $L(s, \Phi)$ and let $\Phi_r(\cdot) = \left(\frac{-r}{\cdot}\right) \Phi(\cdot)$. We denote by $S_k(N, \Phi)$ (resp. $M_k(N, \Phi)$) the space of cusp (resp. modular) forms of weight k, level N, and character $\Phi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. We allow half-integral weight k in the sense of Shimura [Sh1] (though our subscript differs from his).

For $m \in \mathbb{N}$ and a set of primes $\mathbb{S}$, we let $m_{\mathbb{S}} = (m)_{\mathbb{S}}$ denote the product $\prod_{p \in \mathbb{S}} p^{\mathrm{ord}_p(m)}$. We may write $(m)_{\mathfrak{p}|N}$ for $m_{\mathbb{S}}$ where $\mathbb{S}$ is the set of primes dividing N (and similarly for $p \nmid N$).

If G is a group, then we denote its center by $Z(G)$ and commutator subgroup by $[G, G]$. If H is a subgroup of G, we let $N_G(H)$ and $C_G(H)$ respectively denote the normalizer and centralizer of H in G. We say a group G is **quasi-simple** if $G/Z(G)$ is simple and $G = [G, G]$.

If $G = \prod_{i \in I} G_i$ is a direct product of groups G_i, then for every $i \in I$ we define projection maps $\pi_i : G \twoheadrightarrow G_i$. Similarly, for any subset $J \subseteq I$ we define $G_J = \prod_{i \in J} G_i$ and $\pi_J : G \twoheadrightarrow G_J$. We let $Z = Z(G), Z_i = Z(G_i)$, and $Z_J = Z(G_J)$. A **diagonal subgroup** D of G is a subgroup of G such that $\pi_i : D \cong G_i$ for all $i \in I$.

§1 Setting

Throughout this paper, we fix a positive definite integral ternary quadratic form Q, meaning a quadratic form on $\mathbb{Q}^3$ for which $Q(\mathbb{Z}^3) \subseteq \mathbb{Z}$ and $Q(\vec{x}) = 0 \Rightarrow \vec{x} = \vec{0}$. We let D and N respectively denote the determinant and level of Q, and let $\chi(\cdot) = \left(\frac{D}{\cdot}\right)$. We also fix a positive square free integer t and some $T \in t\mathbb{Z}^2$ locally represented by Q, and consider $m = tm_0^2 \in T\mathbb{Z}^2$.

For $f(z) = \sum_{m=1}^\infty a(m)\, \mathbf{e}(mz) \in S_{3/2}(N, \chi)$ and some fixed square free integer $t > 0$ we define its **Shimura lift** $Shi(f) = Shi(f, t)$ to be the modular form $g(z) = \sum_{m=0}^\infty b(m)\, \mathbf{e}(mz) \in M_2(N/2, \chi^2)$ satisfying

$$\sum_{m_0=1}^{\infty} b(m_0)\, m_0^{-s} = L(s, \chi_t) \sum_{m_0=1}^{\infty} a(tm_0^2)\, m_0^{-s}. \tag{1.2}$$

Since χ is quadratic, χ^2 is the trivial character.

Let $U(N,\chi)$ be the subspace of $S_{3/2}(N,\chi)$ spanned by

$$\left\{ u(z) = \sum_{\widetilde{m}\in\mathbb{Z}} \psi(\widetilde{m})\, \widetilde{m}\, \mathbf{e}(t(h\widetilde{m})^2 z) \right\} \tag{1.3}$$

where ψ is a primitive Dirichlet character of conductor R with $\psi(-1) = -1$, ψ_{th^2} agrees with χ on $(\mathbb{Z}/N\mathbb{Z})^\times$ with $4th^2R^2 \mid N$ and $h > 0$. It is known that $U(N,\chi)$ is the subspace of $S_{3/2}(N,\chi)$ whose Shimura lift is not cuspidal. We denote by $U^\perp(N,\chi)$ the subspace of $S_{3/2}(N,\chi)$ perpendicular to $U(N,\chi)$ under the Petersson inner product.

Our approach is to decompose the theta function $\Theta_Q(z)$ as

$$\Theta_Q(z) = E(z) + H(z) + f(z) \tag{1.4}$$

into an Eisenstein series $E(z) = \sum_{m\geq 0} a_E(m)\,\mathbf{e}(mz)$, a spinor term $H(z) \in U(N,\chi)$ with Fourier expansion $\sum_{m>0} a_H(m)\,\mathbf{e}(mz)$ and a cusp form $f(z) \in U^\perp(N,\chi)$. We will also be interested in the form $g = Shi(f) \in S_2(N')$ of level $N' = N/2$ and the primitive character ψ of conductor R above.

From (1.2), we have the relations

$$b(m_0) = \sum_{d\mid m_0,\, d>0} \psi(d)\, a(t(m_0/d)^2) \tag{1.5}$$

$$a(tm_0^2) = \sum_{d\mid m_0,\, d>0} \mu(d)\, \psi(d)\, b(m_0/d) \tag{1.6}$$

To state our main theorems simply, we adopt the following convention for decomposing $H(z)$ and $g(z)$. We write

$$H(z) = \sum_j \delta_j\, u_j(z) \tag{1.7}$$

where $u_j(z) = \sum_{\widetilde{m}\in\mathbb{Z}} \psi(\widetilde{m})\, \widetilde{m}\, \mathbf{e}(t(h_j\widetilde{m})^2 z)$. We also write $g(z) = \sum_{m\geq 0} b(m)\,\mathbf{e}(m)$ as a linear combination of Hecke eigenforms

$$g(z) = \sum_{i=1}^{r} \gamma_i\, g_i(z) \tag{1.8}$$

where $g_i(z) = \sum_{m\geq 0} b_i(m)\,\mathbf{e}(m) = g_i'(d_i z)$ and the $g_i'(z)$ are newforms normalized so that their first Fourier coefficient is 1.

Lemma 1.1. *Let γ_i be as in (1.8). Then $\gamma_i \in \overline{\mathbb{Q}}$.*

Proof. Let K_i be the field of eigenvalues generated by the Fourier coefficients of $g_i(z)$ and $K = \prod_{i=1}^r K_i$. Then the linear form $L(\vec{x}) = \sum_{i=1}^r \gamma_i\, x_i$ is known to vanish on a K-basis from [Sh2', ...], so solving for the γ_i we see that $\gamma_i \in K \subseteq \overline{\mathbb{Q}}$. □

By the theory of newforms [At-Le] and Deligne's bound on Hecke eigenvalues [De], we have that

$$|b_i(m)| \leq \tau(m)\sqrt{m}, \tag{1.9}$$

therefore

$$|b(m)| \leq \tau(m)\sqrt{m} \sum_{i=1}^{r} |\gamma_i| \tag{1.10}$$

and

$$|a(tm_0^2)| \leq \tau(m_0)^2\sqrt{m_0} \sum_{i=1}^{r} |\gamma_i|. \tag{1.10a}$$

If T_0 is locally represented, we say that T_0 is **exceptional** if $a_E(T_0)+a_H(T_0) = 0$ and **anti-exceptional** if $a_E(T_0) = a_H(T_0)$. These are the extremal cases of the general inequality $|a_H(T_0)| \leq a_E(T_0)$ (see [Ha, Lemma 3.4]). We also say that T_0 is **non-exceptional** if it is neither exceptional nor anti-exceptional.

By [Si] and [SP1], in general we have

$$a_E(m) = \frac{\sum_{Q'\in\mathrm{Gen}(Q)} \dfrac{r_{Q'}(m)}{|\mathrm{Aut}(Q')|}}{\sum_{Q'\in\mathrm{Gen}(Q)} \dfrac{1}{|\mathrm{Aut}(Q')|}} \geq 0, \tag{1.11}$$

and similarly

$$a_E(m) + a_H(m) = \frac{\sum_{Q'\in\mathrm{Spn}(Q)} \dfrac{r_{Q'}(m)}{|\mathrm{Aut}(Q')|}}{\sum_{Q'\in\mathrm{Spn}(Q)} \dfrac{1}{|\mathrm{Aut}(Q')|}} \geq 0. \tag{1.12}$$

Since $r_{Q'}(m) \geq 0$, using (1.11) we have

$$0 \leq r_Q(m) \leq |\mathrm{Gen}(Q)|\, a_E(m). \tag{1.13}$$

To discuss the representation behavior of Q within $t\mathbb{Z}^2$ we often write $m_0 = h_k\widetilde{m}$ where h_k is as in (1.7) and for any other h_j there we have $h_j \mid h_k\widetilde{m} \implies h_j \mid h_k$.

Comment. By combining (1.11) and (1.12) we see it is impossible to have both exceptional and anti-exceptional elements in $\mathcal{E}$.

§2 The main term

We now describe the behavior of $r_{Spn(Q)}(m)$ and $r_{Gen(Q)}(m)$ as $m \to \infty$.

Lemma 2.1. *Let $\mathbb{S}$ be a set of primes not dividing N and assume that $m_{\mathbb{S}}$ is $\mathbb{S}$-stable and $ord_p(m_{\mathbb{S}}) = 0$ or 1 for all $p \in \mathbb{S}$. Then*

$$a_E(m_{\mathbb{S}}s^2) = \left(\prod_{p\in\mathbb{S}} \alpha_p\right) a_E(m_{\mathbb{S}})$$

for all $s \in \mathbb{Z}$ divisible only by primes $p \in \mathbb{S}$, where α_p is defined by

$$\alpha_p = \begin{cases} \sum_{\mu=0}^{\nu} p^\mu & \text{if } p \mid m_{\mathbb{S}}, \\ p^\nu & \text{if } p \nmid m_{\mathbb{S}} \text{ and } \psi(p) = 1, \\ p^\nu + 2\sum_{\mu=0}^{\nu-1} p^\mu & \text{if } p \nmid m_{\mathbb{S}} \text{ and } \psi(p) = -1, \end{cases}$$

and $\nu = \nu_p = ord_p(s)$.

Proof. The result follows from [Ha, (3.5) and Table 2], taking $F = \mathbb{Q}$ there. □

Lemma 2.2.

1) If T is non-exceptional then for all $m \in T\mathbb{Z}^2$ we have

$$a_E(m) + a_H(m) \geq C\,(m_0)_{Iso}$$

for some constant $C > 0$.

2) If th^2 is exceptional/anti-exceptional and $m \in th^2\mathbb{Z}^2$ is non-exceptional, then

$$a_E(m) + a_H(m) \geq C\, m_1 \left(\sum_{\substack{d\mid m_{-1} \\ 0<d<m_{-1}}} 2d \right) (m_0)_{p\mid N, Iso},$$

for some constant $C > 0$ depending on h, where $m_0 = h\widetilde{m}$ with $\widetilde{m} = m_1 m_{-1} (m_0)_{p\mid N}$ and m_ε is divisible only by primes $p \nmid N$ with $\psi(p) = \varepsilon = \pm 1$.

Proof. Part 1 is just a weaker statement of [Ha, Thrm 3.7] and is proved there.

For Part 2, we find some prime $p \mid h/m_0$ for which th^2p^2 is non-exceptional and compute the dependency of $a_E(th^2p^2) - |a_H(th^2p^2)|$ on p (with Lemma 2.1). Then since $C_p \geq 1$ for all finitely many isotropic p (see [Ha, Table 2]) our theorem follows after possibly further reducing C to account for possible p-instability of h, primes dividing h, and the finitely many $u_j(z)$ in $H(z)$ as in (1.7).

If $\widetilde{m}$ is only divisible by $p \mid N$ and $p \nmid th^2N$ with $\psi(p) = 1$, since m is represented by Lemma 2.3 we must have $C_p > 1$ for some $p \mid N$. Hence mp^2 is non-exceptional. Otherwise, $\widetilde{m}$ is divisible by some prime q with either $q \mid th^2$ or $\psi(q) = -1$ and in either case mq^2 is not exceptional. □

Lemma 2.3. *Suppose T is locally represented and non-exceptional. Then for all $m = Tm_1^2$ we have*

$$a_E(m) \geq C(m_1)_{Iso},$$

for some constant $C > 0$.

Proof. This is just a weaker statement of [Ha, Cor 3.3] and is proved there. □

§3 Galois representations and twists

In this section we prove a theorem about weight 2 cusp forms whose p^{th} Fourier coefficients are bounded on a certain half of the primes. We do this by analyzing linear combinations of the 2-dimensional l-adic and mod l Galois representations associated to an eigenform as in [Sh2, Ch 7].

Lemma 3.1. *Let ρ_f and ρ_g be the 2-dimensional l-adic Galois representations on $G_{\mathbb{Q}}$ associated to eigenforms $f, g \in S_2(N)$ respectively. Suppose*

$$\rho_f\,|_H = \rho_g\,|_H$$

where $H = Ker(\psi)$ for some non-trivial quadratic character ψ (on $G_{\mathbb{Q}}$) and $l >> 1$. Then either $\rho_g = \rho_f$ or $\rho_g = \rho_f \otimes \psi$.

Proof. Let K be the quadratic extension of $\mathbb{Q}$ associated to ψ, so $H = G_K$. We fix a prime $l >> 1$ and consider the mod l Galois representations

$$\widetilde{\rho}_f, \widetilde{\rho}_g : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_l).$$

We recall the standard definition that the representation ρ on $G_{\mathbb{Q}}$ is CM if there exists an index 2 subgroup H' of $G_{\mathbb{Q}}$ such that $\rho\,|_H$ is abelian.

Case 1: Suppose neither ρ_f nor ρ_g is a CM representation, and consider the product map

$$\widetilde{\rho}_f \times \widetilde{\rho}_g : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_l) \times GL_2(\mathbb{F}_l).$$

From above, we see that the image of H under this map lies in $\Delta = GL_2(\mathbb{F}_l)$ embedded diagonally in the product. By a theorem of Serre [Se, ...] we can choose l sufficiently large so that each

$$\widetilde{\rho}_f, \widetilde{\rho}_g : H \longrightarrow\!\!\!\!\to GL_2(\mathbb{F}_l),$$

which ensures the image of H is the entire diagonal subgroup $\Delta \subset GL_2(\mathbb{F}_l) \times GL_2(\mathbb{F}_l)$.

Since $[G_{\mathbb{Q}} : H] = 2$ we know $[(\widetilde{\rho}_f \times \widetilde{\rho}_g)(G_{\mathbb{Q}}) : \Delta] = 1$ or 2, with the first case giving $\rho_f = \rho_g$. In the second case, consider the map $\widetilde{\rho}_f \times \widetilde{\rho}_g$ composed with the projection isomorphism

$$\pi : (GL_2(\mathbb{F}_l) \times GL_2(\mathbb{F}_l))/\Delta \longrightarrow GL_2(\mathbb{F}_l),$$

whose image gives a group of order 2 in $GL_2(\mathbb{F}_l)$. Since this is abelian, we may form the representation $\widetilde{\rho}_f \cdot \widetilde{\rho}_g^{-1}$ which is a quadratic character on $G_{\mathbb{Q}}/H$, hence $\rho_1 = \rho_2 \otimes \psi$.

Case 2: Suppose that ρ_f is a CM representation (see [R] for details), then there exists a unique imaginary quadratic number field L_f such that ρ_f is abelian on G_{L_f}. This implies (by [R, Prop 4.4]) that ρ_g is also CM since otherwise on $G_{K\cdot L_f}$, ρ_f is abelian but ρ_g is not. In fact, we must have $L_f = L_g$ since otherwise both representations are abelian on $G_K = G_{K\cdot L_f} G_{K\cdot L_g}$, which contradicts the uniqueness of L_f and L_g. Therefore, we may take $L = L_f = L_g$.

Since all such CM forms arise from Hecke characters ξ on L as a sum over integral ideals $\mathfrak{a}$ of L

$$\sum_{\mathfrak{a}} \xi(\mathfrak{a})\,\mathbf{e}(N_{L/\mathbb{Q}}(\mathfrak{a})z)$$

(with $\xi(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not prime to the conductor of ξ), we are saying that $\xi_f = \xi_g$ for all prime ideals $\mathfrak{p}$ of L lying over (all but finitely many of) those $p \in \mathbb{Z}$ with $\psi(p) = 1$. Since each ξ gives rise to a weight 2 form, their infinite parts must have the form $(\frac{u}{|u|})$ (see [I, Thrm 12.5]), so we are reduced to understanding the associated finite characters on some ray class group Λ.

By Tchebetarov's theorem for the ray class field associated to Λ, we know that the group generated by such $Frob_p$ is some subgroup $H \subseteq Gal(L/\mathbb{Q})$ of index ≤ 2 containing $Gal(L/K)$. Since $[L : \mathbb{Q}] = 2$ the index 2 case implies $L = K$. Thus either $\rho_f = \rho_g$ or both f and g have complex multiplication by K. However since the eigenvalues of a CM form for K satisfy $\omega(p) = \psi(p)\omega(p)$ for all but finitely many primes, so by Tchebetarov's theorem the characters of ρ_f and $\rho(g)$ agree on a dense subset of $G_{\mathbb{Q}}$. This together with the continuity of the representations gives $\rho_f = \rho_g$. ☐

Lemma 3.2. *Suppose two l-adic Galois represntations ρ_1 and ρ_2 associated to weight 2 newforms of level N have the same $Tr(\rho_i(Frob_p))$ for all $p \nmid Nl$ with $\psi(p) = \varepsilon = \pm 1$ (where $i = 1, 2$) up to some non-zero constant $\alpha \in \bar{\mathbb{Q}}$, then $\rho_1 = \psi \otimes \rho_2$.*

Proof.

Suppose $\varepsilon = 1$**:** Suppose $Tr(\rho_1(Frob_p)) = \alpha\, Tr(\rho_2(Frob_p))$ for some $\alpha \in \bar{\mathbb{Q}}$ and all $p \nmid Nl$ with $\psi(p) = 1$. By Tchebetarov we see that $Tr(\rho_1) = \alpha\, Tr(\rho_2)$ on G_K. Writing $\alpha = \alpha_2/\alpha_1$ with algebraic integers α_1 and α_2, we have

$$\alpha_1\, Tr(\rho_1|_{G_K}) = \alpha_2\, Tr(\rho_2|_{G_K}).$$

Consider a prime $p \in \mathbb{Z}$ prime to $\alpha_1 \alpha_2$ which splits completely in $\mathbb{Q}(\alpha_1 \alpha_2)$. Then we can find $a_i \in \mathbb{N}$ with $\alpha_i \equiv a_i \pmod{p}$, and we have an equality of the traces of the representations $\oplus_{j=1}^{a_i} \rho_i|_{G_K}$ (where $i = 1$ or 2). By the Brauer-Nesbitt Theorem, theorem we see that these two representations are isomorphic and $\alpha_1 \equiv \alpha_2 \pmod{p}$. Taking $p \to \infty$ we see that $\alpha_1 = \alpha_2$, so $\alpha = 1$ and $\rho_1|_{G_K} \cong \rho_2|_{G_K}$. Using Lemma 3.1 we see $\rho_1 = \psi \otimes \rho_2$.

Suppose $\varepsilon = -1$**:** Suppose $Tr(\rho_1(Frob_p)) = \alpha\, Tr(\rho_2(Frob_p))$ for some $\alpha \in \bar{\mathbb{Q}}$ and all $p \nmid Nl$ with $\psi(p) = -1$. By Tchebetarov we see that $Tr(\rho_1) = \alpha\, Tr(\rho_2)$ on $G_{\mathbb{Q}} - G_K$. Since the ρ_i are known to have complex conjugate eigenvalues λ_p and $\bar{\lambda}_p$ on $Frob_p$, we see that $\lambda_{p,1} = \alpha \lambda_{p,2}$. However $(Frob_p)^3$ is also in $G_{\mathbb{Q}} - G_K$, so we have $\alpha^3 = \alpha$ and $\alpha = \pm 1$. Since $G_{\mathbb{Q}} - G_K$ generates $G_{\mathbb{Q}}$, the ρ_i on $G_{\mathbb{Q}} - G_K$ have unique extensions to $G_{\mathbb{Q}}$. Therefore if $\alpha = 1$ we see $\rho_1 \cong \rho_2$, so $\alpha = -1$. However this is exactly the situation when $\rho_1 = \psi \otimes \rho_2$, so by uniqueness this must be the case. ☐

Lemma 3.3. *Let I be a finite set and D be a subgroup of $\prod_{i \in I} SL_2(\mathbb{F}_l)$ with $l > 3$ prime, and suppose that D projects surjectively onto each factor. Then there exists a unique partition $I = \sqcup_{j=1}^{s} I_j$ and unique diagonal subgroups $D_j \subseteq \prod_{i \in I_j} SL_2(\mathbb{F}_l)$ such that*

$$\prod_{j=1}^{s} D_j \subseteq D \subseteq Z_j \cdot \prod_{j=1}^{s} D_j,$$

where Z_j is the center of $\prod_{i \in I_j} SL_2(\mathbb{F}_l)$.

Proof. Let $G = \prod_{i \in I} G_i$ with $G_i = SL_2(\mathbb{F}_l)$, and for each $x \in G$ let $x_i = \pi_i(x)$. Set $\widetilde{supp}(x) = \{i \in I \mid x_i \notin Z_i\}$. Fix $y \in D - (D \cap Z)$ with $|\widetilde{supp}(y)|$ minimal and set $I_1 = \widetilde{supp}(y)$ and $I^1 = I - I_1$.

Let $D_1^* = D \cap G_{I_1}$ and $D_1 = [D_1^*, D_1^*]$, so that $D_1^* \lhd D, D_1 \lhd D$, and $y \in D_1^*$. For any $i \in I_1$, we have that $\pi_i(D_1^*) \lhd \pi_i(D) = G_i$. Because $y_i \notin Z_i$ we know $\pi_i(D_1^*) \not\subseteq Z_i$, and since $G_i/Z_i = PSL_2(\mathbb{F}_l)$ is simple we have $\pi_i(D_1^*) = G_i$ for all $i \in I_1$.

By our minimal choice of $|\widetilde{supp}(y)|$, we have $D_1^*/(Z \cap D_1^*) \cong G_i/Z_i$ for all $i \in I_1$. Since $G_i = [G_i, G_i]$, $D_1^* = D_1(D_1^* \cap Z)$ with $D_1 = [D_1, D_1]$ and $D_1/(D_1 \cap Z) \cong G_i/Z_i$. Thus D_1 is quasi-simple and $\pi_i(D_1) = \pi_i(D_1^*) = G_i$ for all i. It is known (by Schur multipliers) that for any quasi-simple group Q with $|Z(Q)| = 2$ and $\pi : Q \Rightarrow SL_2(\mathbb{F}_q)$, π is an isomorphism. Hence π_i maps D_1 isomorphically onto G_i for each $i \in I_1$. So D_1 is a diagonal of G_{I_1}.

By projecting D_1 onto some G_i, we see that $N_G(D_1)$ induces only inner automorphisms in D_1 and $C_L(D_1)$ projects into Z_i for all $i \in I_1$. Therefore $N_G(D_1) = (D_1 \times G_{I^1})Z$. As $D_1 \lhd D$, it follows that $D_1 \subseteq \pi_{I_1}(D) \subseteq D_1 Z$ and $DZ = (D_1 \times D^1)Z$ where $D^1 = D \cap G_{I^1}$. In particular for $i \in I^1$, $\pi_i(D^1) = \pi_i(D) = G_i$. The existence of the stated decomposition follows by induction, and the uniqueness is clear. ☐

Theorem 3.4. *Let $\{g_i(z)\}_{i=1}^{r} \subset S_2(N)$ be normalized eigenforms (for all T_p where $p \nmid N$) with Fourier coefficients $b_i(m)$ and ψ be a quadratic Dirichlet character. Suppose there is some constant C such that $|\sum_i \gamma_i\, b_i(p)| \leq C$ for all primes $p \nmid N$ with $\psi(p) = \varepsilon = \pm 1$ and $\gamma_i \in \bar{\mathbb{Q}}$. Then $\sum_i \gamma_i\, b_i(m) = 0$ for all m with $gcd(m, N) = 1$ and $\psi(m) = \varepsilon$.*

Proof. By clearing denominators, we may assume that the γ_i are algebraic integers. Let K_i be the field generated be the Fourier coefficients of g_i, and choose a sequence of primes $l \to \infty$ which split completely in the compositum $K' = \prod K_i$. For each l, let λ be a prime of K' over l, so the residue fields $\mathbb{F}_{\lambda_i} \cong \mathbb{F}_l$ where $\lambda_i = \lambda \cap K_i$.

Consider the mod λ_i Galois representations

$$\rho_i : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_{\lambda_i}) \cong GL_2(\mathbb{F}_l)$$

associated to the g_i, for which $Tr(\rho_i(Frob_p)) \equiv b_i(p) \pmod{\lambda_i}$ for all primes $p \nmid Nl$. With this, we may reinterpret our intial information as a bound on the values attained by the linear form $L(\vec{x}) = \sum_{i=1}^{r} \gamma_i\, x_i$ on the traces of the images $\rho_i(Frob_p)$.

Let K be the quadratic field associated to ψ. By Tchebetarov, we are interested in the image H of the product map $\prod_{i=1}^{r} \rho_i : \sigma G_K \to GL_2(\mathbb{F}_l)^r$ where $\sigma \in G_{\mathbb{Q}}$ with $\psi(\sigma) = \varepsilon$. It is enough for us to analyze the coset G_K since the same arguments will apply to $G_{\mathbb{Q}} - G_K$ by adjusting our linear form by the trace of a coset representative (i.e., replacing γ_i by $\gamma_i\, Tr(\rho_i(\sigma))$).

By a theorem of Serre [Se, ...] each ρ_i is surjective for $l >> 1$, so $\pi_i(H) = GL_2(\mathbb{F}_l)$. Since the commutator subgroup $[GL_2(\mathbb{F}_l), GL_2(\mathbb{F}_l)] = SL_2(\mathbb{F}_l)$, we have $H' := H \cap \prod_{i=1}^{r} SL_2(\mathbb{F}_l) \supseteq [H, H]$ so $\pi_i(H') = \pi_i([H, H]) = SL_2(\mathbb{F}_l)$. By applying the Lemma 3.3 we see that there is a partition $\{1, \cdots, r\} = \sqcup_j I_j$ so that H' breaks up as a direct product $\prod_j Z_j \cdot D_j$ where D_j (resp. Z_j) is a diagonal subgroup (resp. subgroup of the center Z_j) of $\prod_{i \in I_j} SL_2(\mathbb{F}_l)$. Since $Tr(SL_2(\mathbb{F}_l)) = \mathbb{F}_l$, for each j our linear form $L_j(\vec{x}) = \sum_{i \in I_j} \gamma_i\, x_i$ on $Tr(D_j)$ either takes on all values (if $\sum_{i \in I_j} \gamma_i \neq 0$) or is identically zero (otherwise). To analyze role of Z_j, we further partition I_j into subsets I'_k which are maximal for the property: $z \in Z_j \Rightarrow \pi_i(z)$ is constant on I'_k. By considering $z \cdot D_j$ for each $z \in Z_j$ we likewise see (by maximality of the I'_k) that L_j takes on all values on $Tr(Z_j \cdot D_j)$ unless $\sum_{i \in I'_k} \gamma_i = 0$ for every k. This implies that the ρ_i agree on $SL_2(\mathbb{F}_l)$ for all $i \in I'_k$. Since the $g_i(z)$ have the

same level and character, these ρ_i also have the same determinant. Therefore they must be agree on the coset σG_K for all $i \in I'_k$.

Since $|b_i(p)| \leq 2\sqrt{p}$, taking $l \to \infty$ we see that for all $i \in I'_k$ the $b_i(p)$ are equal for any $p \nmid Nl$ with $\psi(p) = \varepsilon$. By considering the associated l-adic Galois representations and using Lemma 3.2, we see that the $g_i(z)$ with $i \in I'_k$ are all twists of each other by ψ. Since $\sum_{i \in I'_k} \gamma_i = 0$, our theorem follows. ☐

Corollary 3.5. *Let $g(z) = \sum_{m=1}^{\infty} b(m)\mathbf{e}(mz) \in S_2(N)$ and ψ a quadratic Dirichlet character. If for some $a \in \mathbb{N}$ we have $|b(aq)| < C$ for all primes q such that $\psi(q) = \varepsilon = \pm 1$, then $b(am) = 0$ for all m with $gcd(m, aN) = 1$ and $\psi(m) = \varepsilon$.*

Proof. By writing $g(z) = \sum_i \gamma_i\, g_i(z)$ as a linear combination of Hecke eigenforms $g_i(z)$ for all T_p with $p \nmid N$, and collecting the $g_i(z) = g'_j(d_j z)$ coming from the same newform $g'_j(z) = \sum_{m=1}^{\infty} b'_j(m)\mathbf{e}(mz)$, we have

$$b(am) = \sum_j \gamma'_j\, b'_j(m)$$

for $gcd(m, aN) = 1$ and some γ'_j. By assumption we also know that $|\sum_j \gamma'_j\, b'_j(q)| \leq C$ for all primes q with $\psi(q) = \varepsilon$, so our result follows from Theorem 3.4. ☐

§4 The exceptional square classes

Theorem 4.1. *If $m = th^2$ is exceptional, then $r_Q(m) = 0$. If m_1 is divisible only by primes $p \nmid N$ with $\psi(p) = 1$, then mm_1^2 is exceptional and $r_Q(mm_1^2) = 0$.*

Proof. Since m is exceptional, (1.12) tells us that $r_Q(th^2) = r_{Spn(Q)}(th^2) = 0$, hence $a(th^2) = 0$. Since td^2 is exceptional for all $d \mid h$, using (1.5) we have $b(h) = 0$. From (1.7) and Lemma 2.1, when $\gcd(m_1, N) = 1$ we know that $t(hm_1)^2$ is exceptional, so replacing h by hm_1 we see that in this case the Lemma holds. ☐

Theorem 4.2. *Let $t\mathbb{Z}^2$ be an exceptional square class for Q. Then for all but finitely many $m \in t\mathbb{Z}^2$ we have*

$$r_{\mathrm{Spn}(Q)}(m) > 0 \implies r_Q(m) > 0,$$

assuming m has a priori bounded divisibility at the anisotropic primes.

Proof. Write $m = tm_0^2$ and $m_0 = h_k \widetilde{m}$ as at the end of §1.

Case 1: *Suppose $\psi(\widetilde{m}) \neq 1$.*

In this case our main term is at least as large as when m is non-exceptional. Comparing Lemma 2.3 with (1.10a) gives a local-global principle when $\widetilde{m} >> 1$, under the usual assumption of bounded divisibility at the anisotropic primes.

Case 2: *Suppose $\psi(\widetilde{m}) = 1$.*

If $\widetilde{m}$ is divisible by some prime q with $\psi(q) = -1$, then from Case 1 we see that $r_{Spn(Q)}(m) > 0$ and $r_Q(th_k^2 q^2) = 0$ for only finitely many q. Thus if $r_Q(m) = 0$ then $\widetilde{m}$ has bounded divisibility at such primes. By comparing Lemma 2.2(2) with (1.10a), we see that a local-global principle holds for $\widetilde{m} >> 1$, under the usual assumption of bounded divisibility at the anisotropic primes. The same asymptotic also gives a local-global principle when $th_k^2(\widetilde{m}^2)_{p|N}$ is not exceptional.

If $th_k^2(\widetilde{m}^2)_{p\nmid N}$ is exceptional and $\widetilde{m}$ is only divisible by primes p with $\psi(p) = 1$, then by Theorem 4.1 we see that $r_{Spn(Q)}(m) = r_Q(m) = 0$, so an exact local-global principle holds. ☐

§5 The anti-exceptional square classes

Theorem 5.1. *Suppose m is anti-exceptional. Then $r_Q(m) = 0 \Rightarrow r_Q(mq^2) = 0$ for all primes $q \nmid mN$ with $\psi(q) = -1$.*

Proof. If $m = tm_0^2$ is anti-exceptional, then so are the td^2 for all $d \mid m_0$ (assuming they are locally represented). Using (1.6), (1.7), and Lemma 2.1 we see

$$r_Q(mq^2) = r_Q(m) + \sum_{d|m_0} \psi(d)\, \mu(d)\, b(m_0 q/d) = \sum_{d|m_0} \psi(d)\, \mu(d)\, b(m_0 q/d).$$

By induction we assume the theorem is true for all $m' < m$ with $m'|m$, thus

$$r_Q(m'q^2) = 0 \qquad \text{and} \qquad r_Q(mq^2) = \pm\, b(m_0 q).$$

From (1.13) we see that $|b(m_0 q)| \leq C$ as q varies. Hence Corollary 3.5 implies $b(m_0 q) = 0$, so $r_Q(mq^2) = 0$. ☐

Theorem 5.2. *Suppose $t\mathbb{Z}^2$ is an anti-exceptional square class for Q. Then there are finitely many sequences mq^2 with m anti-exceptional and q runs over primes $q \nmid N$ with $\psi(q) = -1$ for which a local-global principle fails (i.e., $r_{Spn(Q)}(mq^2) > 0$ but $r_Q(mq^2) = 0$).*

Aside from these sequences, for all but finitely many $m \in t\mathbb{Z}^2$ we have

$$r_{\mathrm{Spn}(Q)}(m) > 0 \implies r_Q(m) > 0,$$

assuming m has a priori bounded divisibility at the anisotropic primes.

Proof. Write $m = tm_0^2$ and $m_0 = h_k \widetilde{m}$ as at the end of §1.

Case 1: *Suppose $\psi(\widetilde{m}) \neq -1$.*

In this case our main term is at least as large as when m is non-exceptional, so by Lemma 2.3 we have a local-global principle when $\widetilde{m} >> 1$ with the usual assumption of bounded divisibility at the anisotropic primes.

Case 2: *Suppose $\psi(\widetilde{m}) = -1$.*

By comparing Lemma 2.2(2) with (1.6) and (1.10), we see that m is represented once $\widetilde{m} >> 1$ unless $m_2 = q$. However in this case, combining Theorem 5.1 with Lemma 2.1 we see that

$$r_{Spn(Q)}(mq^2) > 0 \qquad \text{and} \qquad r_Q(mq^2) = 0$$

for all primes $q \nmid N$ with $\psi(q) = -1$ when m is anti-exceptional and $r_Q(m) = 0$. Hence for these numbers, no local-global principle holds. ☐

§6 An example

We now revisit the example of Schulze-Pillot [SP2] which illustrates the basic behavior we expect within an exceptional square class. It is the genus of forms

$$\begin{aligned} Q1 &= x^2 + 48y^2 + 144z^2 \\ Q2 &= 4x^2 + 48y^2 + 49z^2 + 48yz + 4xz \\ Q3 &= 9x^2 + 16y^2 + 48z^2 \\ Q4 &= 16x^2 + 25y^2 + 25z^2 + 16xz + 16xy + 14yz \end{aligned}$$

which consists of two spinor genera $Spn(Q1) = \{Q1, Q2\}$ and $Spn(Q3) = \{Q3, Q4\}$. The square class $\mathbb{Z}^2$ is exceptional for $Spn(Q3)$ and anti-exceptional for $Spn(Q1)$. We consider the decompositions (1.4) and (1.8) of the theta functions of these forms.

§7 Further Comments

Growth at anisotropic primes.

The results of §4 and §5 characterize the nature of a local-global principle for a ternary form within an exceptional/anti-exceptional square class $t\mathbb{Z}^2$ under the condition of bounded divisibility at the anisotropic primes. To remove this condition, we use the following lemma:

Lemma 7.1. *Suppose p is an anisotropic prime and m is p-stable. Then*

$$r_Q(m) = r_Q(mp^{2\alpha})$$

for all $\alpha \geq 0$.

Proof. In this situation we see from [Ha,(3.16)] that $a_E(m) = a_E(mp^{2\alpha})$. However we know $r_Q(m) \leq r_Q(mp^{2\alpha})$, so together with (1.11) we see $r_Q(m) = r_Q(mp^{2\alpha})$ for all $\alpha \geq 0$. □

From this it suffices to check the p-stability of all locally represented numbers m where $r_Q(m) = 0$ at all anisotropic primes p.

The structure of the cuspidal part $g(z)$.

We also comment on the issue raised by Schulze-Pillot in [SP2] where he notices that the newforms in $g(z)$ for the spinor genera in the example in §6 appear as a sum and difference of a form and its twist by $\left(\frac{-3}{\cdot}\right)$ respectively.

As a consequence of the proof of Theorem 3.4, we have shown that in general if $m = th^2$ is exceptional/anti-exceptional then the cusp forms $g_i(z)$ with $d_i = h$ appear in pairs as a difference/sum of it and its twist by ψ respectively. This shows that Schulze-Pillot's observation occurs in every exceptional/anti-exceptional situation, and it is this which allows us to understand the representation behavior within such square classes.

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