Probabilistic and Stochastic perspectives on instantaneously rebalanced constant-weighted portfolio strategies in an antisymmetric two stock market

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1 Introduction

This paper is an attempt to understand the meaning and behavior of an "instantaneously rebalanced" constant-weighted portfolio in a market consisting of two stocks whose price movements are perfectly anti-correlated and driven by a single log-Brownian process.

In the real world any portfolio strategy is implemented by rebalancing/trading the portfolio at a discrete set of times t_i in between which the portfolio evolves according to market price movements. Mathematically this can be described as a sequence of "buy-and-hold" portfolio strategies occurring on closed time intervals $[t_i, t_{i+1}]$ that partition the given time interval $[T_0, T_1]$ on which the portfolio exists. Here the portfolio holdings (i.e. the amount of each asset held in the portfolio) are well-defined away from the discrete set of rebalancing times t_i, but the total portfolio value is well-defined on the entire interval $[T_0, T_1]$.

For the more idealized notion of an "instantaneously rebalanced" portfolio, the portfolio holdings at any given time are not well-defined, and instead one describes how its total (log-) value process evolves in time according to some notion of "infinitesimal price fluctuations" that must be clearly specified and requires some justification.

One model for these infinitesimal price fluctuations is given by Stochastic Portfolio Theory, which assumes a continuous semi-martingale market and uses the Itô stochastic differential formalism to describe the stochastic differential $d\log(Z_{\vec{\pi}}(t))$ of the portfolio log-value process. This provides a quick way of computing the asymptotic growth rate of many portfolios via Itô's rule, but its infinitesimal meaning is somewhat hidden in the L^2-closure used to define the Itô integral which makes it hard to give more than a heuristic proof for why it accurately describes the behavior of portfolios on arbitrarily small timescales. Another more naive approach to describe instantaneous rebalancing is to follow the limiting-type definition used to interpret the classical differentials of Calculus, and to take the limit of buy-and-hold portfolios on arbitrarily fine partitions of a given time interval.

In this paper we develop the mathematics of this second limiting-style probabilistic perspective and use it to determine the log-value process of any instantaneously rebalanced constant-weighted portfolio strategy. Along the way we also describe various related structures, tools and techniques (e.g. an unnormalized Law of Large Numbers (LLN), asymptotic series, Laplace's method, quadratic approximation) that are useful for working with this perspective more generally. We also compare the results of these two perspectives and find that they give the same value processes in all cases. This provides a concrete interpretation of and greater insight into the formalism of stochastic calculus, and also shows that the stochastic notion of an instantaneously rebalanced portfolio is well-approximated by any real world implementation of the strategy provided that the portfolio strategy rebalancing times are sufficiently close together.

2 Basic definitions

Our main object of study is any given "instantaneously rebalanced" constant-weighted portfolio strategy in the log-normal antisymmetric two stock market $\mathcal{M}$. We begin by giving some basic definitions used to describe these portfolio strategies and their performance over time.

Definition 1 (The antisymmetric market $\mathcal{M}$)**.** *We define* **the antisymmetric market** $\mathcal{M}$ *to be the stochastic log-price process* $\vec{r}(t) := (r_1(t), r_2(t)) := (y_t, -y_t) \in \mathbb{R}^2$ *where* $y_t := W_t$ *is a standard Brownian motion for* $t \in [0, \infty)$ *(i.e. with mean zero and variance* t *at time* t*). Here the price (in dollars) of the* i*-th stock at time* t *is given by* $e^{r_i(t)}$*, and we consider each stock as having only one "share" outstanding.*

Definition 2 (Portfolios)**.** *We define a* **portfolio** *at time* t *for* $\mathcal{M}$ *to be any (possibly random) vector* $\vec{\pi} := \vec{\pi}(t) := (\pi_1, \pi_2) := (\pi_1(t), \pi_2(t)) \in \mathbb{R}^2$ *whose components* $\pi_i(t)$ *reflect the value (in dollars) held in the* i*-th stock of* $\mathcal{M}$ *at time* t*. The* **value** *of* $\vec{\pi}$ *at time* t *is defined as* $\Sigma\vec{\pi}(t) = \pi_1(t) + \pi_2(t)$ *and the* **log-value** *of* $\vec{\pi}$ *at time* t *is defined as* $\ln(\Sigma\vec{\pi}(t))$ *when* $\Sigma\vec{\pi}(t) > 0$*.*

Remark 3 (Portolio strategies and SPT)**.** *To be compatible with the language of Stochastic Portfolio Theory (SPT), we can also define the portfolio strategy associated to a portfolio* $\vec{\pi}$ *as the normalized vector* $\vec{\pi}/\Sigma\vec{\pi}$ *whose components are called the* **portfolio weights** *(assuming* $\Sigma\vec{\pi}(t) > 0$*). In SPT this vector of weights is what is referred to as a portfolio. When considering the value process* $Z_{\vec{\pi}}(t)$ *of a SPT portfolio* $\vec{\pi}$ *with* $\Sigma\vec{\pi} = 1$*, its initial value at time* $t_0 = 0$ *is usually taken to be 1. We will call any vector* $\vec{w} = (w_1, w_2) \in \mathbb{R}^2$ *satisfying* $w_1 + w_2 = 1$ *a* **weight vector***.*

Definition 4 (Buy-and-hold portfolio strategies)**.** *Given an initial time* t_0*, an initial value* $V_0 \in \mathbb{R} > 0$*, and a (weight) vector* $\vec{w} = (w_1, w_2) \in \mathbb{R}^2$ *with* $w_1 + w_2 = 1$*, then we say that the* **buy-and-hold portfolio strategy** $\mathrm{BH}_{\vec{w}}$ **with weight** $\vec{w}$ **and initial value** V_0 *in the antisymmetric two stock market* $\mathcal{M}$ *is given by the process*

$$\vec{\pi}_{\mathrm{BH}_{\vec{w},V_0}}(t) := V_0 \cdot \vec{w} \cdot e^{\vec{r}(t)-\vec{r}(t_0)} := \left(V_0 w_1 e^{r_1(t)-r_1(t_0)}, V_0 w_2 e^{r_2(t)-r_2(t_0)}\right)$$

for any time $t \geq t_0$*. For simplicity, when* $V_0 = 1$ *we omit it from the notation. If a final time* $t_1 \geq t_0$ *is also given, then we restict to* $t \in [t_0, t_1]$ *and say that* $\vec{\pi}(t_1)$ *is the* **final portfolio***, otherwise we take* $t_1 := \infty$*. We also let*

$$Z_{\mathrm{BH}_{\vec{w},V_0},[t_0,t_1]}(t) := \Sigma\vec{\pi}_{\mathrm{BH}_{\vec{w},V_0}}(t) = V_0 \cdot \Sigma\vec{\pi}_{\mathrm{BH}_{\vec{w}}}(t)$$

denote the **value process** *of this portfolio for all* $t \in [t_0, t_1]$*.*

Definition 5 (Discrete portfolio strategies)**.** *We say that a* **discrete portfolio strategy** S *for* $\mathcal{M}$ *on* $[a, b]$ *is a portfolio process* $\vec{\pi}_S(t)$ *that for some partition* $a = t_0 < t_1 < \cdots < t_r = b$ *of* $[a, b]$ *and some (weight) vectors* $\vec{w}_j$ *where* $\Sigma\vec{w}_j = 1$ *for all* $0 \leq j \leq r-1$ *has the form*

$$\vec{\pi}_S(t) = (\Sigma\vec{\pi}_S(t_j)) \cdot \vec{\pi}_{\mathrm{BH}_{\vec{w}_j}}(t)$$

when $t \in [t_j, t_{j+1}]$. *While* $\vec{\pi}_S(t)$ *is not well-defined at the partition points* t_j, *its* **value process** $Z_S(t) := \Sigma\vec{\pi}_S(t)$ *is well-defined for all* $t \in [a, b]$.

Definition 6 (Approximately constant-weighted strategies)**.** *We define the* **approximately constant-weighted portfolio strategy** *with weight vector* $\vec{w}$ *and duration* Δt *to be the discrete portfolio strategy where for all* $0 \leq j \leq n$ *we have* $\vec{w}_j = \vec{w}$ *and* $t_{j+1} - t_j = \Delta t$. *When* $\vec{w} = (\frac{1}{2}, \frac{1}{2})$ *we refer to this as the* **approximately equal-weighted portfolio strategy***.*

It is very convenient to discuss a portfolio's performance in terms of its log-returns, and to notice that this quantity is scale invariant.

Definition 7 (Strategy log-returns)**.** *Given a discrete portfolio strategy* S *on a time interval* $[t_0, t_1]$ *with positive initial value* $V_0 = \Sigma\vec{\pi}_S(t_0) > 0$, *we define the* **log-returns** *for this strategy on* $[t_0, t_1]$ *to be the difference*

$$\mathrm{LogRet}(S, V_0, [t_0, t_1]) := \ln(\Sigma\vec{\pi}_S(t_1)) - \ln(\Sigma\vec{\pi}_S(t_0)).$$

When $\mathrm{LogRet}(S, V_0, [t_0, t_1])$ *is independent of the initial value* V_0 *we write it more concisely as* $\mathrm{LogRet}(S, [t_0, t_1])$.

Lemma 8 (Log-return scale invariance)**.** *The log-return of any discrete portfolio strategy* S *is independent of its (positive) initial value, i.e. for any* $\lambda \in \mathbb{R}_{>0}$ *we have* $\mathrm{LogRet}(S, \lambda V_0, [t_0, t_1]) = \mathrm{LogRet}(S, V_0, [t_0, t_1])$, *so the notation* $\mathrm{LogRet}(S, [t_0, t_1])$ *is meaningful.*

Proof. This follows directly from the observation that scaling the initial portfolio by λ also scales the final portfolio by λ, which is clear from the definition of the buy-and-hold strategies on each time interval $[t_j, t_{j+1}]$. □

Our main purpose is to study the change in value over time of an "instantaneously rebalanced" constant-weighted portfolio strategy, which we now define via its limiting log-value process.

Definition 9 (Constant-weighted portfolio strategy)**.** *We say a portfolio process* $\vec{\pi}(t)$ *on the closed time interval* $[T_0, T_1]$ *is a* **constant-weighted portfolio strategy** $\mathrm{CW}_{\vec{w}}$ *with weight vector* $\vec{w}$ *and initial portfolio value* $V_0 = \Sigma\vec{\pi}(T_0)$ *if for every* $T \in [T_0, T_1]$ *the log-returns on* $[T_0, T]$ *are defined as the limit in probability of the random variables*

$$\mathrm{LogRet}(\mathrm{CW}_{\vec{w}}, [T_0, T]) := \lim_{\substack{n\to\infty \\ prob.}} \sum_{i=0}^{n-1} \mathrm{LogRet}(\mathrm{BH}_{\vec{w}}, [t_i, t_{i+1}])$$

where $t_i := T_0 + \frac{i(T-T_0)}{n}$ *for all* $0 \leq i \leq n$, *and the portfolio weights are given by the constant vector* $\frac{\vec{\pi}(T)}{\Sigma\vec{\pi}(T)} = \vec{w}$. *When* $\vec{w} = (\frac{1}{2}, \frac{1}{2})$ *we refer to this as the* **equal-weighted portfolio strategy** $\mathrm{EW} := \mathrm{CW}_{(\frac{1}{2},\frac{1}{2})}$.

Remark 10 (Notions of convergence for partions)**.** *Our choice to use the weaker notion of convergence in probability instead of almost sure convergence in Definition 9 is more convenient but also loses little in terms of precision since more generally one would be interested in considering a limit over all partitions* $\mathcal{P} : T_0 = t_0 < t_1 < \cdots < t_n = T_1$ *of* $[T_0, T_1]$ *and then allowing the maximal subinterval size* $\|\mathcal{P}\| := \max\{|t_{i+1} - t_i| \mid 0 \leq i < n\}$ *to approach zero. However convergence in probability of a sequence of random variables implies almost sure convergence of a subsequence (to the same limit), so we could simply take this subsequence of partitions to obtain almost sure convergence.*

The difference between these two notions of convergence is a certain uniformity of how the limit is approached, but our main concern is with the limit itself as opposed to the kind of convergence used to attain it.

Remark 11 (Usefulness of scale-invariance)**.** *The scale-invariance of the buy-and-hold portfolio log-returns shown in Lemma 8 allows us to ignore the value of the approximately equal-weighted portfolios at each starting time* t_i*, and instead just accumulate the log-returns with initial portfolio* $\vec{\pi}(T_0) = \vec{w}$ *and* $\Sigma\vec{w} = 1$*.*

3 The equal-weighted portfolio strategy

In this section we analyze the log-value of the equal-weighted portfolio strategy in the antisymmetric market $\mathcal{M}$. We do this first from the perspective of Stochastic Portfolio Theory (SPT) and then directly from the definition of the equal-weighted portfolio strategy.

3.1 The equal-weighted portfolio SPT log-value

In the language of Stochastic Portfolio Theory as described in [2], the antisymmetric market $\mathcal{M}$ with two stocks can be thought of as a market with parameters $n = 2$, $\vec{\gamma}(t) = \vec{0}$, and $\Xi(t) = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right]$. From this perspective we can describe the equal-weighted portfolio log-return random variable $\log(Z_{\vec{\pi}}(t))$ at time t either directly from [2, Prop 1.1.5, p6] or as a functionally generated portfolio via [2, Thrm 3.1.5, p46] generated by $S(\vec{x}) := (x_1 x_2)^{1/2}$. From either perspective we have the following result.

Theorem 12 (SPT log-return random variable)**.** *Suppose that* $\vec{\pi}(t)$ *is the SPT instantaneously rebalanced equal-weighted portfolio in the antisymmetric two stock market* $\mathcal{M}$ *with initial value* $Z_{\vec{\pi}}(0) = 1$*. Then the portfolio log-return random variable* $\log(Z_{\vec{\pi}}(t))$ *at every time* $t \geq 0$ *is (a.s.) given by the deterministic random variable*

$$\log(Z_{\vec{\pi}}(t)) = \gamma_{\vec{\pi}}(t) = \tfrac{1}{2}t.$$

Proof. From [2, Prop 1.1.5, p6] we know that the stochastic process $\{\log(Z_{\vec{\pi}}(t))\}_{t\geq 0}$ satisfies the stochastic differential equation

$$d\log(Z_{\vec{\pi}}(t)) = \gamma_{\vec{\pi}}(t)\,dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t)$$

where in our case we have $n = 2$, $\pi_i(t) = \frac{1}{2}$ and $\Xi(t) := (\xi_{i\nu}) = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right]$. Substituting these gives the simplified expression

$$d\log(Z_{\vec{\pi}}(t)) = \gamma_{\vec{\pi}}(t)\,dt$$

which we evaluate by computing the growth rate process $\vec{\gamma}(t) = \vec{0}$ and covariance process $\sigma(t) := (\sigma_{ij}(t))_{1\leq i,j\leq n} := \Xi\,\Xi^T = \left[\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right]$. This gives

$$\begin{aligned} \gamma_{\vec{\pi}}(t) &= \sum_{i=1}^{2} \pi_i(t)\gamma_i(t) + \frac{1}{2}\left[\sum_{i=1}^{2} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{2} \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right] \\ &= 0 + \frac{1}{2}\left[\left(\frac{1}{2} + \frac{1}{2}\right) - 0\right] = \frac{1}{2}, \end{aligned}$$

which together with the initial condition $\log Z_{\vec{\pi}}(0) = 0$ proves the theorem since $\int_0^t \frac{1}{2}\,ds = \frac{t}{2}$. □

3.2 Approximately equal-weighted portfolios

From the perspective of the Gaussian transition densities that define the evolution of Brownian motion on any given time interval, we can also evaluate the equal-weighted portfolio log-return random variable explicitly as a limit of buy-and-hold portfolios held repeatedly for a duration Δt after which they are rebalanced, as $\Delta t \to 0$. This is the approximately equal-weighted portfolio strategy with duration Δt. To understand it we begin by analyzing the behavior of the equal-weighted buy-and-hold portfolio strategy in the antisymmetric market $\mathcal{M}$.

Lemma 13 (Buy-and-hold log-returns)**.** *Suppose that the initial portfolio $\vec{\pi} = (\frac{1}{2}, \frac{1}{2})$ evolves as a buy-and-hold portfolio in the antisymmetric market $\mathcal{M}$ over a time interval $[t_0, t_1]$. Then its log-returns are given by the random variable*

$$\mathrm{LogRet}(\mathrm{BH}_{(\frac{1}{2},\frac{1}{2})}, [t_0, t_1]) = \ln(\cosh(r))$$

where $r := \mathcal{N}(0, \Delta t)$ and $\Delta t := t_1 - t_0$. Further its expected log-returns are given by

$$\mathbb{E}\left[\mathrm{LogRet}(\mathrm{BH}_{(\frac{1}{2},\frac{1}{2})}, [t_0, t_1])\right] = \frac{1}{\sqrt{2\pi\Delta t}} \int_{-\infty}^{\infty} \ln(\cosh(x))\, e^{\frac{-x^2}{2\Delta t}}\,dx.$$

Proof. This follows from the definition of the buy-and-hold strategy final portfolio as $\vec{\pi}(t_1) := \left(\frac{e^r}{2}, \frac{1}{2e^r}\right)$, and the definitions of LogRet and $\cosh(x)$. The expected value expression follows from the known probability density function of the normal distribution $\mathcal{N}(0, \Delta t)$. □

Remark 14 (Equal-weighted portfolio growth picture)**.** *By looking at the graph of the function*

$$f(r) := \ln\left(\frac{e^r}{2} + \frac{1}{2e^r}\right) = -\ln(2) + \ln\left(e^r + e^{-r}\right)$$

it becomes clear why the equal-weighted portfolio profits from market volatility, since any deviation from $r = 0$ will increase the log-value of the portfolio (i.e. $r = 0$ is a global minimum of $f(r)$), thereby increasing the portfolio value.

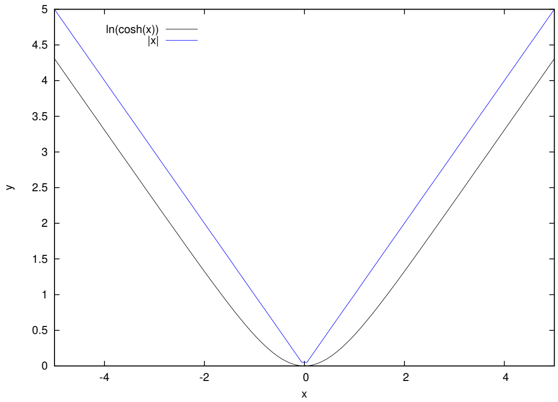

Figure 1: Plots of $\ln(\cosh(x))$ and $|x|$ as functions of x.

Next we state a version of the Law of Large Numbers that will be useful for understanding the performance of the equal-weighted portfolio.

Lemma 15 (Unnormalized LLN)**.** *Let $r_n := \mathcal{N}(0, \frac{T}{n})$, suppose $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $0 \le f(x) \le |x|^b$ for any $b \in \mathbb{R}_{>1}$, and for each $n \in \mathbb{N}$ suppose that the independent random variables $X_{n,i}$ for all $1 \le i \le n$ have the same distribution as $f(r_n)$. Then we have the L^2-convergence*

$$\lim_{\substack{n\to\infty \\ L^2}} \sum_{i=1}^{n} X_{n,i} = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}[X_{n,i}],$$

hence also convergence in probability, assuming that one of the limits exist.

Proof. For each $n \in \mathbb{N}$ we consider the centered random variables $Y_{n,i} := X_{n,i} - \mathbb{E}\left[X_{n,i}\right]$, $Y_n := \sum_{i=1}^{n} Y_{n,i}$, and compute the variance

$$\mathbb{E}\left[(Y_n)^2\right] = \mathbb{E}\left[\left(\sum_{i=1}^{n} Y_{n,i}\right)^2\right]$$

$$= \sum_{i=1}^{n} \mathbb{E}\left[(Y_{n,i})^2\right] + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbb{E}\left[Y_{n,i}\, Y_{n,j}\right]$$

$$\overset{indep.}{=} \sum_{i=1}^{n} \mathbb{E}\left[(Y_{n,i})^2\right] + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \underbrace{\mathbb{E}\left[Y_{n,i}\right]}_{=0} \cdot \underbrace{\mathbb{E}\left[Y_{n,j}\right]}_{=0}$$

$$\overset{i.d.}{=} n \cdot \mathbb{E}\left[(Y_{n,1})^2\right] + 0$$

$$= n \cdot \mathbb{E}\left[(X_{n,1} - \mathbb{E}\left[X_{n,1}\right])^2\right]$$

$$= n \cdot \mathbb{E}\left[(X_{n,1})^2\right] - n \cdot (\mathbb{E}\left[X_{n,1}\right])^2$$

$$\leq n \cdot \mathbb{E}\left[(X_{n,1})^2\right]$$

$$\leq n \cdot \mathbb{E}[|r_n|^{2b}].$$

This together with the known absolute moment formula

$$\mathbb{E}\left[|\mathcal{N}(0,\sigma^2)|^p\right] = \sigma^p \left(\frac{2}{\pi}\right)^{\frac{p}{2}} \Gamma\left(\frac{p+1}{2}\right) \qquad \text{for all } p \in \mathbb{R} > -1$$

gives the inequality

$$0 \leq \mathbb{E}[(Y_n)^2] \leq n^{1-b} \cdot \left(\frac{2T}{\pi}\right)^b \cdot \Gamma\left(\frac{2b+1}{2}\right)$$

which shows that $\{Y_n\} \overset{L^2}{\to} 0$ as $n \to \infty$. This proves the desired L^2-convergence, which implies convergence in probability (e.g. see [1, Thrm 4.1.4, p71]). □

Lemma 16 (Equal-weighted LLN Bound)**.** *We have the inequalities*

$$0 \leq \ln(\cosh(x)) \leq |x|^b$$

for every $b \in \mathbb{R}$ satisfying $1 \leq b \leq 2$.

Proof. This is equivalent to showing that $\cosh(x) \leq e^{|x|^b}$ for all $1 \leq b \leq 2$ for all $x \in \mathbb{R}$. When $b = 1$ this follows from the inequality

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \leq e^{|x|},$$

and when $b = 2$ this follows from noting that the Taylor expansion of the function

$$f(x) = e^{x^2} - \ln(\cosh(x)) = \sum_{i=1}^{\infty} \left(\frac{1}{i!} - \frac{1}{(2i)!}\right) x^{2i}$$

converges for all $x \in \mathbb{R}$ and has only positive coefficients, so $f(x) \geq 0$. For $1 \leq b \leq 2$ we have the inequalities

$$0 \leq \ln(\cosh(x)) \leq |x| \leq |x|^b \qquad \text{when } |x| \geq 1,$$

$$0 \le \ln(\cosh(x)) \le |x|^2 \le |x|^b \qquad \text{when } |x| \le 1,$$

which together prove the lemma. □

With this lemma we can show that the instantaneously rebalanced equal-weighted strategy log-returns are given by a deterministic random variable whose linearity in time follows from simple invariance and scaling properties.

Theorem 17 (Equal-weighted log-returns)**.** *The log-return of the equal-weighted portfolio strategy* EW *in the antisymmetric market $\mathcal{M}$ is given by the deterministic random variable*

$$\mathrm{LogRet}(\mathrm{EW}, [T_0, T_1]) = \frac{d}{dt^+}\left(\mathbb{E}[\mathrm{LogRet}(\mathrm{BH}_{(\frac{1}{2},\frac{1}{2})}, [0, (T_1 - T_0)t])]\right|_{t=0}.$$

Proof. From the definition, we have that

$$\mathrm{LogRet}(\mathrm{EW}, [T_0, T_1]) = \lim_{\substack{n\to\infty \\ prob.}} \sum_{i=0}^{n-1} \mathrm{LogRet}(\mathrm{BH}_{(\frac{1}{2},\frac{1}{2})}, [t_i, t_{i+1}])$$

where $t_i := T_0 + \frac{i(T_1-T_0)}{n}$. By Lemma 16 and since $\mathcal{M}$ has stationary, independent increments, we can apply Lemma 15 to see that this is almost surely equal to the number

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=0}^{n-1} \mathbb{E}\left[\mathrm{LogRet}(\mathrm{BH}_{(\frac{1}{2},\frac{1}{2})}, [t_i, t_{i+1}])\right] \\
&\qquad = \lim_{n\to\infty} n\mathbb{E}\left[\mathrm{LogRet}(\mathrm{BH}_{(\frac{1}{2},\frac{1}{2})}, [0, \tfrac{T_1-T_0}{n}])\right] \\
&\qquad \overset{t:=\frac{1}{n}}{=} \lim_{t\to 0^+} \frac{\mathbb{E}\left[\mathrm{LogRet}(\mathrm{BH}_{(\frac{1}{2},\frac{1}{2})}, [0, (T_1 - T_0)t])\right]}{t} \\
&\qquad = \frac{d}{dt^+}\left(\mathbb{E}\left[\mathrm{LogRet}(\mathrm{BH}_{(\frac{1}{2},\frac{1}{2})}, [0, (T_1 - T_0)t])\right]\right|_{t=0}
\end{aligned}$$

which proves the theorem. □

Lemma 18 (Equal-weighted linearity in time)**.** *The log-returns of the equal-weighted portfolio strategy in the antisymmetric market $\mathcal{M}$ satisfies*

$$\mathrm{LogRet}(\mathrm{EW}, [T_0, T_1]) \overset{dist.}{=} (T_1 - T_0) \cdot \mathrm{LogRet}(\mathrm{EW}, [0, 1])$$

as an equality of random variables in distribution.

Proof. By Theorem 17 we know that $\mathrm{LogRet}(\mathrm{EW}, [T_0, T_1])$ is a deterministic random variable. Since $\mathcal{M}$ has stationary increments, we know that

$$\mathrm{LogRet}(\mathrm{EW}, [T_0, T_1]) \overset{dist.}{=} \mathrm{LogRet}(\mathrm{EW}, [0, T_1 - T_0])$$

when $T_1 \geq T_0$ and that

$$\text{LogRet}(\text{EW}, [0,T]) \overset{dist.}{=} \sum_{i=1}^{n} \text{LogRet}(\text{EW}, [0, \tfrac{T}{n}]) \overset{dist.}{=} n \cdot \text{LogRet}(\text{EW}, [0, \tfrac{T}{n}])$$

for all $n \in \mathbb{N}$ when $T \geq 0$. By taking $T \in \mathbb{N}$ we see that for all rational $T \in \mathbb{R}_{>0}$ we have that $\text{LogRet}(\text{EW}, [0,T]) \overset{dist.}{=} cT$ where $c := \text{LogRet}(\text{EW}, [0,1])$.

Now assume (by contradiction) that there is some irrational $T' \in \mathbb{R}_{>0}$ with $\text{LogRet}(\text{EW}, [0,T']) = bT' \neq cT'$ for some $b \in \mathbb{R}$. By scaling T' we may assume that $|cT' - bT'| > M$ for any arbitrarily large $M \in \mathbb{R}_{>0}$, and since the rational numbers are dense in $\mathbb{R}$ for any $\varepsilon > 0$ we can choose a rational number T so that $|T - T'| < \varepsilon$ and cT is between bT' and cT'. Therefore $|cT - bT'| > |cT' - bT'|$ and

$$|\text{LogRet}(\text{EW}[0, T - T'])| = |cT - bT'| > |cT' - bT'| > M,$$

which implies that there are arbitrarily small T'' at which $|\text{LogRet}(\text{EW}, [0,T''])|$ is arbitrarily large, so $\lim_{T'' \to 0^+, dist.} \text{LogRet}(\text{EW}, [0,T''])$ does not exist.

However from Definition 9 and deterministicness we see that

$$\begin{aligned}
\lim_{\substack{T \to 0^+ \\ dist.}} \text{LogRet}(\text{EW}, [0,T]) &\overset{dist.}{=} \lim_{T \to 0^+} \mathbb{E}[\text{LogRet}(\text{EW}, [0,T])] \\
&= \lim_{T \to 0^+} \mathbb{E}\left[\lim_{\substack{n \to \infty \\ a.s.}} \sum_{i=0}^{n-1} \text{LogRet}(\text{BH}_{(\frac{1}{2},\frac{1}{2})}, [t_i, t_{i+1}])\right] \\
&= \lim_{T \to 0^+} \lim_{n \to \infty} \sum_{i=1}^{n} \mathbb{E}\left[\text{LogRet}(\text{BH}_{(\frac{1}{2},\frac{1}{2})}, [0, \tfrac{T}{n}])\right] \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{\lim_{T \to 0^+} \mathbb{E}\left[\text{LogRet}(\text{BH}_{(\frac{1}{2},\frac{1}{2})}, [0, \tfrac{T}{n}])\right]}_{=0} \\
&= 0
\end{aligned}$$

This contradiction shows that $\text{LogRet}(\text{EW}, [0,T]) = cT$ for all $T \in \mathbb{R}_{>0}$, proving the lemma. ☐

Theorem 19 (Simplified equal-weighted log-returns)**.** *The log-returns of the equal-weighted portfolio strategy in the antisymmetric market $\mathcal{M}$ satisfies*

$$\text{LogRet}(\text{EW}, [T_0, T_1]) = (T_1 - T_0) \cdot \frac{d}{dt^+} \left(\mathbb{E}[\text{LogRet}(\text{BH}_{(\frac{1}{2},\frac{1}{2})}, [0,t])]\right|_{t=0}.$$

Proof. This follows from using the linearity in Lemma 18 to replace $[T_0, T_1]$ by $[0,1]$, then applying the formula in Theorem 17. ☐

3.3 Naive evaluation by Laplace's method

By using Laplace's method we can determine an asymptotic formula for the behavior of the equal-weighted buy-and-hold expected log-returns over small time intervals. With some care, this allows us to evaluate the log-return process of the instantaneously rebalanced equal-weighted strategy.

Definition 20 (Asymptotic equality)**.** *Given two functions $f(x)$ and $g(x)$ which are non-zero in an open punctured neighborhood $U = U - \{a\}$ of some $a \in \mathbb{R}$. Then we say that $f(x)$ and $g(x)$ are* **asymptotically equal** *if $\frac{f(x)}{g(x)} \to 1$ as $x \to a$, denoted by $f(x) \sim g(x)$ as $x \to a$. The analogous definitions apply to one-sided limits, and allowing $a \in \{\pm\infty\}$.*

Lemma 21 (Asymptotic expected value as $t \to 0^+$)**.** *The expected log-returns of the buy-and-hold portfolio in the antisymmetric market $\mathcal{M}$ over a small time interval $[0, t]$ is asymptotically given by*

$$\mathbb{E}\left[\mathrm{LogRet}(\mathrm{BH}_{(\frac{1}{2},\frac{1}{2})}, [0, t])\right] \sim \frac{t}{2}$$

as $t \to 0^+$.

Proof. Laplace's method allows us to identify asymptotics of integrals of the form $I(u) := \int_a^b f(x)e^{u \cdot g(x)}\, dx$ as $u \to \infty$, which we take to be $f(x) := \ln(\cosh(x))$ and $g(x) = -\frac{x^2}{2}$ where $u := \frac{1}{t}$. Here we $x = 0$ as the maximum of $g(x)$, so we have

$$I(u) \sim \sqrt{\frac{u}{2\pi}} \int_{-\varepsilon}^{\varepsilon} \ln(\cosh(x))\, e^{\frac{-u \cdot x^2}{2}}\, dx$$

Next we use the standard three step process for estimating this integral via the Taylor expansion of $f(x)$ at $x = 0$, giving

$$\begin{aligned}
I(u) &\sim \frac{1}{\sqrt{2\pi u^{-1}}} \int_{-\varepsilon}^{\varepsilon} \ln(\cosh(x))\, e^{\frac{-x^2}{2u^{-1}}}\, dx && \text{(Interval restriction)}\\
&\sim \frac{1}{\sqrt{2\pi u^{-1}}} \int_{-\varepsilon}^{\varepsilon} \frac{x^2}{2}\, e^{\frac{-x^2}{2u^{-1}}}\, dx && \text{(Taylor approximation)}\\
&\sim \frac{1}{\sqrt{2\pi u^{-1}}} \int_{-\infty}^{\infty} \frac{x^2}{2}\, e^{\frac{-x^2}{2u^{-1}}}\, dx && \text{(Interval expansion)}\\
&= \frac{1}{2}\mathbb{E}_{\mathcal{N}(0,u^{-1})}[X^2] = \frac{1}{2u}.
\end{aligned}$$

(For the interested reader, these estimates are justified carefully in Lemmas 25, 26, and 27.) Writing this in terms of $t = \frac{1}{u}$ proves the assertion. □

Lemma 22 (Expected right-derivative at $t = 0$)**.** *The right-derivative of the expected log-returns of a buy-and-hold portfolio in the antisymmetric market $\mathcal{M}$ is given by*

$$\frac{d}{dt^+}\left(\mathbb{E}[\mathrm{LogRet}(\mathrm{BH}_{(\frac{1}{2},\frac{1}{2})},[0,t])]\right|_{t=0} = \frac{1}{2}.$$

Proof. This follows directly from the asymptotic equality in Lemma 21. □

Remark 23 (Right-derivative by L'Hopital's rule)**.** *If one is simply interested in the statement of Lemma 22 then it can be derived more simply by applying L'Hopital's rule as*

$$\begin{aligned}
\frac{d}{dt^+}\left(\mathbb{E}[\mathrm{LogRet}(\mathrm{BH}_{(\frac{1}{2},\frac{1}{2})},[0,t])]\right|_{t=0} &= \lim_{t\to 0^+}\frac{\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}\ln(\cosh(x))e^{\frac{-x^2}{2t}}\,dx}{t}\\
&\overset{(\sigma:=\sqrt{t})}{=} \lim_{\sigma\to 0^+}\frac{\frac{1}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}}\ln(\cosh(x))e^{\frac{-x^2}{2\sigma^2}}\,dx}{\sigma^2}\\
&\overset{(y:=\frac{x}{\sigma})}{=} \lim_{\sigma\to 0^+}\frac{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\ln(\cosh(y\sigma))e^{\frac{-y^2}{2}}\,dy}{\sigma^2}\\
&\overset{(L'Hopital)}{=} \lim_{\sigma\to 0^+}\frac{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\tanh(y\sigma)ye^{\frac{-y^2}{2}}\,dy}{2\sigma}\\
&\overset{(L'Hopital)}{=} \lim_{\sigma\to 0^+}\frac{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{sech}^2(y\sigma)y^2e^{\frac{-y^2}{2}}\,dy}{2}\\
&= \frac{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}y^2e^{\frac{-y^2}{2}}\,dy}{2} = \frac{1}{2}
\end{aligned}$$

since the exponential decay of the integral justifies differentiation under the integral sign.

Corollary 24 (Equal-weighted portfolio value)**.** *The log-returns of an instantaneously rebalanced equal-weighted portfolio in the antisymmetric $\mathcal{M}$ over the time interval $[T_0, T_1]$ is given by*

$$\mathrm{LogRet}(\mathrm{EW},[T_0,T_1]) = \frac{T_1 - T_0}{2}.$$

Proof. This follows directly from Theorem 19 and Lemma 22. □

3.4 Laplace's method error estimates

To justify the rough calculation in Lemma 21 we now give a series of explicit error estimates arising from Laplace's method that will help us to understand the accuracy of its predicted asymptotic.

Lemma 25 (Interval restriction)**.** *If we let*

$$E_1^+(u,\varepsilon) := \sqrt{\frac{u}{2\pi}} \int_\varepsilon^\infty \ln(\cosh(x))\, e^{\frac{-ux^2}{2}}\, dx,$$

then

$$0 < E_1^+(u,\varepsilon) \leq \frac{e^{\frac{-u\varepsilon^2}{2}}}{\sqrt{2\pi u}}.$$

Proof. By using the upper bound

$$\ln(\cosh(x)) = \ln\left(\frac{e^x + e^{-x}}{2}\right) \leq \ln\left(\frac{e^{|x|} + e^{|x|}}{2}\right) = \ln\left(e^{|x|}\right) = |x|,$$

we have that

$$\begin{aligned} E_1^+(u,\varepsilon) &= \sqrt{\frac{u}{2\pi}} \int_\varepsilon^\infty \ln(\cosh(x))\, e^{\frac{-ux^2}{2}}\, dx \\ &\leq \sqrt{\frac{u}{2\pi}} \int_\varepsilon^\infty x\, e^{\frac{-ux^2}{2}}\, dx \\ &= \sqrt{\frac{u}{2\pi}} \left(\frac{-1}{u} \cdot e^{\frac{-ux^2}{2}} \right|_\varepsilon^\infty \\ &= \frac{-1}{\sqrt{2\pi u}} \left(0 - e^{\frac{-u\varepsilon^2}{2}} \right|_\varepsilon^\infty \\ &= \frac{e^{\frac{-u\varepsilon^2}{2}}}{\sqrt{2\pi u}}. \end{aligned}$$

This proves the upper bound, and the lower bound follows from the non-negativity of the integrand. □

Lemma 26 (Taylor approximation)**.** *If we let*

$$E_2(u,\varepsilon;k) := \sqrt{\frac{u}{2\pi}} \int_{-\varepsilon}^\varepsilon R_k(x) e^{\frac{-ux^2}{2}}\, dx$$

where $R_k(x)$ is the error term for the degree k Taylor approximation of the function $f(x) := \ln(\cosh(x))$ near $x = 0$, then

$$E_2(u,\varepsilon;k) \leq \frac{M_{\varepsilon,k+1}}{(k+1)!} \cdot \frac{\Gamma\left(\frac{k+2}{2}\right)}{\sqrt{2\pi}} \cdot \frac{1}{u^{\frac{k+1}{2}}}$$

where $M_{\varepsilon,k+1}$ is the maximum value of the $(k+1)$-st derivative of $\ln(\cosh(x))$ on the closed interval $[-\varepsilon,\varepsilon]$.

Proof. By applying the Lagrange mean-value form of the remainder term in Taylor's theorem, we know that $R_k(x) = \frac{f^{(k+1)}(\xi_x)}{(k+1)!} x^{k+1}$ for some $\xi_x \in [0, x]$ depending on x. Therefore we have the upper bound

$$\begin{aligned}
E_2(u, \varepsilon; k) &= \sqrt{\frac{u}{2\pi}} \int_{-\varepsilon}^{\varepsilon} R_k(x) e^{\frac{-ux^2}{2}}\, dx \\
&\leq \sqrt{\frac{u}{2\pi}} \int_{-\varepsilon}^{\varepsilon} \frac{M_{\varepsilon,k+1}}{(k+1)!} |x|^{k+1} e^{\frac{-ux^2}{2}}\, dx \\
&= \sqrt{\frac{u}{2\pi}} \frac{M_{\varepsilon,k+1}}{(k+1)!} \int_{-\varepsilon}^{\varepsilon} |x|^{k+1} e^{\frac{-ux^2}{2}}\, dx \\
&\leq \sqrt{\frac{u}{2\pi}} \frac{M_{\varepsilon,k+1}}{(k+1)!} \int_{-\infty}^{\infty} |x|^{k+1} e^{\frac{-ux^2}{2}}\, dx \\
&= \sqrt{\frac{u}{2\pi}} \frac{M_{\varepsilon,k+1}}{(k+1)!} \frac{\Gamma\left(\frac{k+2}{2}\right)}{u^{\frac{k+2}{2}}}
\end{aligned}$$

which proves the theorem. □

Lemma 27 (Interval expansion)**.** *If we let*

$$E_3^+(u, \varepsilon) := \sqrt{\frac{u}{2\pi}} \int_{\varepsilon}^{\infty} \tfrac{x^2}{2}\, e^{\frac{-ux^2}{2}}\, dx,$$

then when $u > 1$ we have the upper bound

$$E_3^+(u, \varepsilon) < \frac{Ce^{-Au}}{\sqrt{u}}$$

for some explicit constants $A, C \in \mathbb{R}_{>0}$ depending only on ε.

Proof. By making the change of variables $y := x - \varepsilon$ we have that

$$\begin{aligned}
E_3^+(u, \varepsilon) &= \sqrt{\frac{u}{2\pi}} \int_{\varepsilon}^{\infty} \tfrac{x^2}{2}\, e^{\frac{-ux^2}{2}}\, dx \\
&= \sqrt{\frac{u}{2\pi}} \int_{0}^{\infty} \tfrac{(y+\varepsilon)^2}{2}\, e^{\frac{-u(y+\varepsilon)^2}{2}}\, dy \\
&= \sqrt{\frac{u}{2\pi}} e^{\frac{-u\varepsilon^2}{2}} \int_{0}^{\infty} \tfrac{(y+\varepsilon)^2}{2}\, e^{\frac{-u(y^2+2y\varepsilon)}{2}}\, dy \\
&\leq \sqrt{\frac{u}{2\pi}} \frac{e^{\frac{-u\varepsilon^2}{2}}}{2} \int_{0}^{\infty} (y+\varepsilon)^2\, e^{-uy\varepsilon}\, dy \\
&= \sqrt{\frac{u}{2\pi}} \frac{e^{\frac{-u\varepsilon^2}{2}}}{2} \int_{0}^{\infty} (y^2 + 2y\varepsilon + \varepsilon^2)\, e^{-uy\varepsilon}\, dy \\
&= \sqrt{\frac{u}{2\pi}} \frac{e^{\frac{-u\varepsilon^2}{2}}}{2} \left[\frac{2!}{(u\varepsilon)^3} + \frac{1! \cdot 2\varepsilon}{(u\varepsilon)^2} + \frac{0! \cdot \varepsilon^2}{(u\varepsilon)^1}\right]
\end{aligned}$$

where the last formula results from the known integration-by-parts identity $\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$. When $u > 1$ we see that $u^{-1} \geq u^{-k}$ for any $k \geq 1$, so we have the upper bound

$$E_3^+(u,\varepsilon) \leq \sqrt{\frac{u}{2\pi}}\frac{e^{\frac{-u\varepsilon^2}{2}}}{2}\left[\frac{2!}{(u\varepsilon)^3} + \frac{1!\cdot 2\varepsilon}{(u\varepsilon)^2} + \frac{0!\cdot\varepsilon^2}{(u\varepsilon)^1}\right]$$
$$\leq \sqrt{\frac{u}{2\pi}}\frac{e^{\frac{-u\varepsilon^2}{2}}}{2u}\left[2\varepsilon^{-3} + 2\varepsilon^{-1} + \varepsilon\right]$$

which proves the theorem. □

3.5 The excess growth power series

To the reader familiar with the theory of power series, it may appear that the calculation of the previous subsections is overly complicated since it is well-known that the linear term of a power series expansion at zero is given by the derivative of the function at zero. Unfortunately this reasoning does not apply in our situation since the power series has zero radius of convergence.

Definition 28 (Excess growth series)**.** *We define the* **excess growth power series** *for the equal-weighted portfolio strategy in the antisymmetric market* $\mathcal{M}$ *to be the formal power series*

$$EG_{\mathrm{EW},\mathcal{M}}(t) := \mathbb{E}_{\mathcal{N}(0,t)}[\ln(\cosh(x))] := \sum_{i=0}^{\infty}\mathbb{E}_{\mathcal{N}(0,t)}[c_i x^i]$$

in t *obtained by computing the term-by-term expected values of the convergent power series* $\ln(\cosh(x)) =: \sum_{i=0}^{\infty} c_i x^i$*. We also define its degree* D *Taylor polynomials by*

$$EG_{\mathrm{EW},\mathcal{M},\leq D}(t) := \sum_{i=0}^{D}\mathbb{E}_{\mathcal{N}(0,t)}[c_i x^i].$$

Theorem 29 (Exact excess growth expected values)**.** *The excess growth power series is given by*

$$EG_{\mathrm{EW},\mathcal{M}}(t) := \sum_{D:=2D'\in 2\mathbb{N}} \frac{2^{D'}(2^D-1)B_D}{(D')!\cdot D}\cdot(\Delta t)^{D'}$$

where B_n *is the* n*-th Bernoulli number.*

Proof. From term-by-term integration of the known power series for $\tanh(x)$ and the fact that $\ln(\cosh(0)) = 0$ we have the power series expansion

$$\ln(\cosh(x)) = \sum_{D\in 2\mathbb{N}} \frac{2^D(2^D-1)B_D}{(D)!\cdot D}x^D$$

which converges absolutely when $x \in (\frac{-\pi}{2}, \frac{\pi}{2})$. This gives the expected values

$$\mathbb{E}_{\mathcal{N}(0,t)}[c_D x^D] = (\Delta t)^{D'} \cdot \frac{\cancel{(2D')!}}{(D')! \cdot \cancel{2^{D'}}} \cdot \frac{\cancel{2^{D'}} 2^{D'}(2^D - 1)B_D}{\cancel{D!} \cdot D}$$
$$= (\Delta t)^{D'} \cdot \frac{2^{D'}(2^D - 1)B_D}{(D')! \cdot D},$$

which proves the theorem. □

This explicit combinatorial description of the formal excess growth power series makes it clear that its radius of convergence is zero, so it is not meaningful as a power series description of the equal-weighted portfolio log-returns for small times $t > 0$.

Corollary 30 (Divergence of expected-log-value)**.** *The excess growth power series $EG_{\mathrm{EW},\mathcal{M}}(t)$ has zero radius of convergence.*

Proof. We apply the ratio test to compute the radius of convergence $\rho := \lim_{n\to\infty} \frac{c_n}{c_{n+1}}$ of the power series $\sum_{n=0}^{\infty} c_n x^n$ where

$$c_n := \frac{2^{D'}(2^D - 1)B_D}{(D')! \cdot D}$$

By the asymptotic formula

$$|B_{2n}| \sim 4\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n}$$

for Bernoulli numbers, we see that

$$c_n \sim \frac{2^n(2^{2n} - 1)}{(n!) \cdot 2n} \cdot 4\sqrt{\pi n} \cdot \left(\frac{n}{\pi e}\right)^{2n}.$$

as $n \to \infty$. Therefore as $n \to \infty$ we have that

$$\frac{c_n}{c_{n+1}} \sim \frac{2^n(2^{2n} - 1)}{(n!) \cdot 2n} \cdot 4\sqrt{\pi n} \cdot \left(\frac{n}{\pi e}\right)^{2n} \cdot \frac{(n+1)! \cdot (2n+2)}{2^{n+1}(2^{2n+2} - 1)} \cdot \frac{1}{4\sqrt{\pi(n+1)}} \cdot \left(\frac{\pi e}{n+1}\right)^{2n+2}$$
$$\sim \frac{(n+1)\cancel{(2n+2)}}{2 \cdot \cancel{(2n)} \cdot 4} \cdot \cancel{\sqrt{\frac{n}{n+1}}} \cdot \frac{(\pi e)^2 (n)^{2n}}{(n+1)^{2n+2}}$$
$$\sim \frac{(n+1)(\pi e)^2}{8 \cdot n^2} \left(\frac{n}{n+1}\right)^{2n+2}$$
$$\sim \frac{\cancel{(n+1)}(\pi e)^2}{8 \cdot n^{\cancel{2}}} \left(\left(1 - \frac{1}{n+1}\right)^{n+1}\right)^2 \sim \frac{(\pi \cancel{e})^2}{8 \cdot n} \cdot \cancel{e^{-2}} \sim \frac{\pi^2}{8n}$$

which shows that $\rho = \lim_{n\to\infty} \frac{\pi^2}{8n} = 0$, proving the claim. □

Remark 31 (Conditions for excess growth series convergence). *If we considered a function defined by a power series $f(x) = \sum_{i=0}^{\infty} c_i x^i$ where $|c_{n+1}| > K|c_n|n$ for some $K > 0$ when n is sufficiently large, then the formal excess growth power series $\mathbb{E}_{\mathcal{N}(0,t)}[f(x)]$ for $f(x)$ will have a non-zero radius of convergence. (This is easily seen from the ratio test.)*

Example 32 (Exact excess growth expected value terms). *From Theorem 29 we see the first nine terms of the excess growth power series are given by*

$$EG_{\mathrm{EW},\mathcal{M},\leq 9}(t) = \frac{t}{2} - \frac{t^2}{4} + \frac{t^3}{3} - \frac{17t^4}{24} + \frac{31t^5}{15} - \frac{691t^6}{90} + \frac{10922t^7}{315} - \frac{929569t^8}{5040} + \frac{3202291t^9}{2835}.$$

3.6 Asymptotic series

While the formal excess growth power series is not very useful as a convergent power series because it is divergent, it can still be regarded as giving a good approximation of the expected log-return for the buy-and-hold portfolio for small times $t > 0$ as an asymptotic series.

Definition 33 (Asymptotically smaller). *Given two real-valued functions $f(x)$ and $g(x)$ defined in an open neighborhood U of $a \in \mathbb{R}$, we say that $f(x)$ is* **asymptotically smaller** *than $g(x)$ as $x \to a$, and write $g(x) = o(f(x))$, if for every $\varepsilon > 0$ there is some $\delta := \delta(\varepsilon) > 0$ so that $|g(x)| < \varepsilon |f(x)|$ when $|x - a| < \delta$. By slight abuse of notation, we also allow $a \in \{\pm\infty\}$.*

Definition 34 (Asymptotic series). *Given a function $f(x)$ on an interval $I \subseteq \mathbb{R}$ a sequence of functions $\phi_i(x)$ defined on I, and some $a \in I$, we say that the series $\sum_{i=0}^{\infty} c_i \phi_i(x)$ is an* **asymptotic series** *for $f(x)$ near a if for every $N > 0$ we have that*

$$f(x) = \sum_{i=0}^{N} c_i \phi_i(x) + o(\phi_N(x))$$

as $x \to a$. By slight abuse of notation, we also allow $a \in \{\pm\infty\}$.

Here we are interested in two kinds of asymptotic series, power series in x as $x \to 0^+$ where $\phi_i(x) := x^i$, and power series in $\frac{1}{x}$ as $x \to \infty$ where $\phi_i(x) := x^{-i}$.

Theorem 35 (Excess growth asymptotic series). *The equal-weighted excess growth power series is an asymptotic series expansion for $\mathbb{E}_{\mathcal{N}(0,t)}[\ln(\cosh(x))]$ as $t \to 0^+$.*

Proof. To see that we obtain an asymptotic series, we compute the effect of taking the degree $k := 2k'$ Taylor approximation to $\ln(\cosh(x))$ at $x = 0$ and compare this to the error incurred from using Laplace's method to estimate the expected value.

Using the degree k Taylor approximation for $\ln(\cosh(x))$ gives that

$$\begin{aligned}\mathbb{E}_{\mathcal{N}(0,t)}[\ln(\cosh(x))] &= EG_{\mathrm{EW},\mathcal{M},\leq k'}(t) + \mathrm{Err}_k(t)\\ &= EG_{\mathrm{EW},\mathcal{M},\leq k'}(t) + \mathrm{Err}_{k+1}(t)\end{aligned}$$

where $EG_{\mathrm{EW},\mathcal{M},\leq k'}(t)$ is a polynomial of degree k' where $\mathrm{Err}_{k+1}(t)$ denotes the error from taking the degree $(k+1)$ Taylor approximation to $\ln(\cosh(x))$ and applying Laplace's method. This error can be further broken down into its contributions respectively from interval restriction to $(-\varepsilon, \varepsilon)$, Taylor approximation, and interval extension to $(-\infty, \infty)$ as

$$\mathrm{Err}_{k+1}(t) = 2E_1^+(u, \varepsilon) + E_2(u, \varepsilon; k+1) + 2E_3^+(u, \varepsilon; k+1)$$

where $t := 1/u$ and any fixed $\varepsilon > 0$. From Lemmas 25 and 26 we see that $2E_1^+(u, \varepsilon) = o(e^{\frac{-u\varepsilon^2}{2}})$ and $E_2(u, \varepsilon; k+1) = o(u^{\frac{-(k+2-\kappa)}{2}})$ for any $\kappa > 0$ as $u \to \infty$. By replacing $\frac{x^2}{2}$ with the degree k Taylor approximation to $\ln(\cosh(x))$ in Lemma 27 and its proof, we see that also $2E_3^+(u, \varepsilon) = o(e^{\frac{-u\varepsilon^2}{2}})$ as $u \to \infty$. Therefore combining these shows that

$$\mathrm{Err}_{k+1}(t) = o(e^{\frac{-u\varepsilon^2}{2}}) + o(u^{\frac{-(k+2-\kappa)}{2}}) + o(e^{\frac{-u\varepsilon^2}{2}}) = o(u^{\frac{-k}{2}})$$

as $u \to \infty$ and so $\mathrm{Err}_{k+1}(t) = o(t^{\frac{k}{2}}) = o(t^{k'})$ as $t \to 0^+$, proving the theorem. □

3.7 Taylor polynomials as good asymptotic approximations

The following picture illustrates the nature and accuracy of the asymptotic series approximation given by the equal-weighted excess growth series. Since this asymptotic series is divergent as a power series, the length of the interval $[0, \varepsilon)$ over which it gives a good approximation shrinks to zero as the degree of the approximating Taylor polynomial increases.

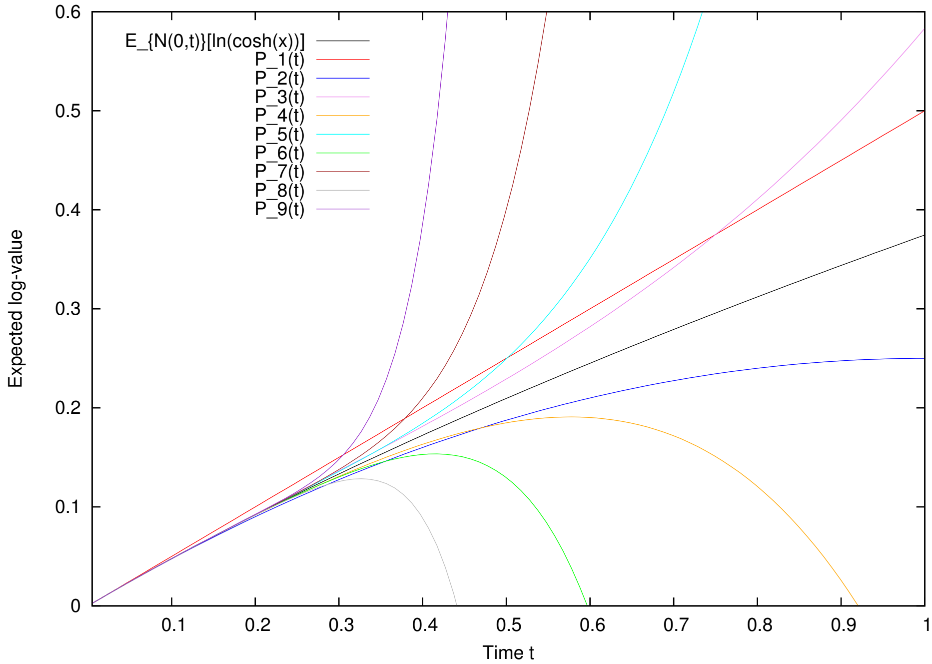

Figure 2: Plot of $\mathbb{E}_{\mathcal{N}(0,t)}\left[\ln(\cosh(x))\right]$ as a function of time t together with its first nine Taylor polynomials $P_i(t)$ for $1 \leq i \leq 9$, as given in Example 32.

4 Constant-weighted portfolio strategies

In the previous section we took care to analyze many aspects of the equal-weighted portfolio strategy, which is perhaps the simplest example of an instantaneously rebalanced portfolio strategy. In this section we'll look at slightly more general constant-weighted portfolio strategies, which somewhat surprisingly require a different and more sophisticated approach to understand probabilistically.

4.1 Constant-weighted portfolio SPT log-values

Just as with the equal-weighted portfolio, we can use [2, Prop 1.1.5, p6] to describe the SPT log-value process $\log(Z_{\vec{\pi}(t)})$ for the constant-weighted portfolio with portfolio weights $\frac{\vec{\pi}}{\Sigma\vec{\pi}} = (w_1, w_2)$.

Theorem 36 (SPT log-returns)**.** *Suppose that $\vec{\pi}(t)$ is the constant-weighted portfolio in the antisymmetric market $\mathcal{M}$ with initial value $Z_{\vec{\pi}}(0) = 1$. Then the SPT log-value at time $t \geq 0$ is (a.s.) given by the random variable*

$$\log(Z_{\vec{\pi}}(t)) = \tfrac{1-(w_1-w_2)^2}{2}t + (w_1 - w_2)W_t,$$

where W_t is the standard Brownian motion. This is a deterministic random variable $\iff w_1 = w_2$.

Proof. As in Theorem 12 we have that $[\xi_{i\nu}] = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right]$ and $[\sigma_{ij}] = \left[\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right]$ and $\{\log(Z_{\vec{\pi}}(t))\}_{t\geq 0}$ satisfies the stochastic differential equation

$$d\log(Z_{\vec{\pi}}(t)) = \gamma_{\vec{\pi}}(t)\,dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t).$$

Here $n = 2$ and all $\gamma_i(t) = 0$, which gives

$$\begin{aligned}\gamma_{\vec{\pi}}(t) &= \sum_{i=1}^{2} \pi_i(t)\gamma_i(t) + \frac{1}{2}\left[\sum_{i=1}^{2} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{2} \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right] \\ &= 0 + \frac{1}{2}\left[\underbrace{(w_1 + w_2)}_{=1} - (w_1^2 - 2w_1w_2 + w_2^2)\right] = \frac{1-(w_1-w_2)^2}{2}.\end{aligned}$$

This shows that the SPT log-value Itô differential is

$$d\log(Z_{\vec{\pi}}(t)) = \tfrac{1-(w_1-w_2)^2}{2}dt + (w_1 - w_2)dW_t$$

which together with its initial value of zero integrates to give

$$\log(Z_{\vec{\pi}}(t)) = \tfrac{1-(w_1-w_2)^2}{2}t + (w_1 - w_2)W_t,$$

proving the theorem. □

4.2 Quadratic approximation for Gaussian-driven processes

In order to carry out the evaluation of our main limit for approximately constant-weighted portfolios with unequal weights we will need to take a different approach, since the tangent line to the funtion

$$f(r) := \mathrm{LogRet}(\mathrm{BH}_{(w_1,w_2)}, [T_0, T_1])$$

at $r = 0$ is not horizontal. Instead we show that for any function $g(r)$ that admits an asymptotic series expansion about $r = 0$ we can approximate the desired limit by replacing $g(r)$ by its quadratic Taylor approximation, and that this recovers the usual Itô calculus rules for a change of variables.

The main tools we need to perform this approximation are Chebyshev's inequality and estimates for the moments of the normal distribution, and this effect can be thought of a concrete manifestation of the general "concentration of measure" phenomenon.

Lemma 37 (Weak concentration of measure)**.** *Suppose that* $\{X_{n,i}\}_{n\in\mathbb{N},1\leq i\leq n}$ *is a family of i.i.d. random variables with* $X_{n,i} \overset{dist.}{=} \mathcal{N}(0, \frac{\sigma^2}{n})$ *and let*

$$S_{n;k} := \sum_{i=1}^{n}(X_{n,i})^k.$$

Then for every $k \in \mathbb{N} > 2$ *and every* $\varepsilon \in \mathbb{R} > 0$ *we have that*

$$\lim_{n\to\infty} \mathbb{P}(|S_{n;k}| \geq \varepsilon) = 0.$$

Proof. By applying the Chebyshev inequality to $S_{n;k}$ we have that

$$\mathbb{P}(|S_{n,k} - \mathbb{E}[S_{n,k}]| \geq \varepsilon) \leq \frac{\mathrm{Var}(S_{n,k})}{\varepsilon^2},$$

and also the variance $\mathrm{Var}(S_{n;k})$ can be computed explicitly in terms of the Gaussian moments as

$$\mathrm{Var}(S_{n;k}) = \sum_{i=1}^{n} \mathrm{Var}((X_{n,i})^k) = n\cdot \mathrm{Var}((X_{n,1})^k) = n\cdot \mathbb{E}[\mathcal{N}(0, \tfrac{\sigma^2}{n})^{2k}] = \tfrac{C_{2k}\sigma^{2k}}{n^{k-1}}.$$

for some constants C_k depending on k (where $C_k = 0$ when k is odd). When $k > 1$ this shows that

$$\lim_{n\to\infty} \mathrm{Var}(S_{n;k}) = 0. \tag{1}$$

We can also compute the expected value

$$\mathbb{E}[S_{n,k}] = \mathbb{E}\left[\sum_{i=1}^{n}(X_{n,i})^k\right] = \sum_{i=1}^{n}\mathbb{E}[(X_{n,i})^k] = n\mathbb{E}[(X_{n,i})^k] = n\cdot \tfrac{C_k}{\sqrt{n}^k}$$

which shows that $\lim_{n\to\infty} \mathbb{E}[S_{n,k}] = 0$ when $k > 2$ and therefore that

$$\lim_{n\to\infty} \mathbb{P}(|S_{n,k} - \mathbb{E}[S_{n,k}]| \geq \varepsilon) = \lim_{n\to\infty} \mathbb{P}(|S_{n,k}| \geq \varepsilon). \tag{2}$$

The lemma follows by combining equations (1) and (2). □

Lemma 38 (Gaussian tail vanishing)**.** *Suppose that* $g(x)$ *is a function on* $\mathbb{R}$ *and* $X_n \overset{dist.}{=} \mathcal{N}(0, \frac{\sigma^2}{n})$ *is a random variable so that for some* $\varepsilon > 0$ *and for any sufficiently large* $n \in \mathbb{N}$ *the expected value* $\mathbb{E}[g(X_n)\cdot \mathbb{1}_{|X_n|>\varepsilon}]$ *exists. Then*

$$\lim_{n\to\infty} n\cdot \mathbb{E}[g(X_n)\cdot \mathbb{1}_{|X_n|>\varepsilon}] = 0.$$

Proof. Let $s_n := n \cdot \mathbb{E}[g(X_n)\cdot \mathbb{1}_{|X_n|>\varepsilon}]$ for all $n \in \mathbb{N}$. We will show that the sequence $(s_n)_{n\in\mathbb{N}}$ approaches zero as $n \to \infty$ by showing that there is some $M > 0$ and some positive real number $K < 1$ so that

$$n > M \implies s_{n+1} \leq K \cdot s_n. \tag{3}$$

From this and the finiteness of some element s_N, we see that when $n > M$ we also have $s_{n+k} \leq s_n \cdot K^k$, so $\lim_{k\to\infty} |s_{n+k}| \leq \lim_{k\to\infty} |s_n| \cdot K^k = 0$.

To see that (3) holds we compute an upper bound on the ratio

$$R_n(x) := \frac{(n+1)\cdot \cancel{g(x)} \cdot \rho_{X_{n+1}}(x)}{n \cdot \cancel{g(x)} \cdot \rho_{X_n}(x)}$$

when $|x| > \varepsilon$ and n is sufficiently large. We have that

$$R_n(x) = \frac{n+1}{n} \cdot \frac{\sqrt{\frac{n+1}{\cancel{2\pi\sigma^2}}} \cdot \left(e^{\frac{-x^2}{\sigma^2}}\right)^{\cancel{n}+1}}{\sqrt{\frac{n}{\cancel{2\pi\sigma^2}}} \cdot \cancel{\left(e^{\frac{-x^2}{\sigma^2}}\right)^{n}}} = \left(\frac{n+1}{n}\right)^{\frac{3}{2}} \cdot e^{\frac{-x^2}{\sigma^2}},$$

so $R_n(x) \leq \left(\frac{n+1}{n}\right)^{\frac{3}{2}} \cdot e^{\frac{-\varepsilon^2}{\sigma^2}}$ when $|x| \geq \varepsilon$ and also $0 \leq \lim_{n\to\infty} R_n(x) \leq e^{\frac{-\varepsilon^2}{\sigma^2}} =: K$ uniformly in x. This shows that for sufficiently large n we have

$$\begin{aligned} s_{n+1} &= (n+1) \int_{|x|>\varepsilon} g(x) \cdot \rho_{X_{n+1}}(x)\, dx \\ &= n \int_{|x|>\varepsilon} R_n(x) \cdot g(x) \cdot \rho_{X_n}(x)\, dx \\ &\leq K \cdot n \int_{|x|>\varepsilon} g(x) \cdot \rho_{X_n}(x)\, dx = K \cdot s_n \end{aligned}$$

with $|K| < 1$, proving that the desired limit has the form

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} C \cdot K^n = 0.$$

□

Lemma 39 (Expected value vanishing)**.** *Suppose that* $g : \mathbb{R} \to \mathbb{R}$ *satisfies* $g(x) = o(x^2)$ *as* $x \to 0$, $\{X_{n,i}\}_{n\in\mathbb{N}, 1\leq i\leq n}$ *is a family of i.i.d. random variables with* $X_{n,i} \overset{dist.}{=} \mathcal{N}(0, \frac{\sigma^2}{n})$*, and let*

$$S_n := \sum_{i=1}^{n} g(X_{n,i}).$$

Then

$$\lim_{n\to\infty} \mathbb{E}[S_n] = 0.$$

Proof. From Lemma 38 in the limit we can neglect the behavior of $g(x)$ outside any fixed open neighborhood of zero, and so we can adjust $g(x)$ so that for any given $\varepsilon > 0$ we have that $g(x) \leq \varepsilon x^2$ for all $x \in \mathbb{R}$. We also assume w.l.o.g. that $g(x) = |g(x)|$ is non-negative, since the result in this case and for $-g(x)$ implies the result for all $g(x)$ by the squeeze theorem. However this means that for any $\varepsilon > 0$ and sufficiently large n we have that

$$\begin{aligned} 0 \leq \mathbb{E}[S_n] &= \sum_{i=1}^{n} \mathbb{E}[g(X_{n,i})] = n \cdot \mathbb{E}[g(X_{n,1})] \\ &\leq n \cdot \varepsilon \cdot \mathrm{Var}(X_{n,1}) = \cancel{n} \cdot \varepsilon \cdot \frac{\sigma^2}{\cancel{n}} = \varepsilon \cdot \sigma^2, \end{aligned}$$

so therefore $\lim_{n\to\infty} \mathbb{E}[S_n] = 0$ as desired. □

Lemma 40 (Variance vanishing)**.** *Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ satisfies $g(x) = o(x^1)$ as $x \to 0$, $\{X_{n,i}\}_{n\in\mathbb{N},1\leq i\leq n}$ is a family of i.i.d. random variables with $X_{n,i} \stackrel{dist.}{=} \mathcal{N}(0, \frac{\sigma^2}{n})$, and let*

$$S_n := \sum_{i=1}^{n} g(X_{n,i}).$$

Then

$$\lim_{n\to\infty} \mathrm{Var}[S_n] = 0.$$

Proof. By the same reasoning as in the proof of Lemma 39, for any given $\varepsilon > 0$ we can replace $g(x)$ by a function so that $g(x) = |g(x)| \leq \varepsilon|x|$ for all $x \in \mathbb{R}$ without affecting the limit. In this case for any $\varepsilon > 0$ and sufficiently large n we have that

$$\begin{aligned} 0 \leq \mathrm{Var}[S_n] &= \sum_{i=1}^{n} \mathrm{Var}[g(X_{n,i})] = n \cdot \mathrm{Var}[g(X_{n,1})] \\ &= n \cdot \left[\mathbb{E}[g(X_{n,1})^2] - (\mathbb{E}[g(X_{n,1})])^2\right] \leq n \cdot \mathbb{E}[g(X_{n,1})^2] \\ &\leq n \cdot \varepsilon^2 \cdot \mathrm{Var}(X_{n,1}) = \cancel{n} \cdot \varepsilon^2 \cdot \frac{\sigma^2}{\cancel{n}} = \varepsilon^2 \cdot \sigma^2, \end{aligned}$$

so therefore $\lim_{n\to\infty} \mathrm{Var}[S_n] = 0$ as desired. □

Theorem 41 (Concentration of measure)**.** *Suppose that $\{X_{n,i}\}_{n\in\mathbb{N},1\leq i\leq n}$ is a family of i.i.d. random variables with $X_{n,i} \stackrel{dist.}{=} \mathcal{N}(0, \frac{\sigma^2}{n})$, $g : \mathbb{R} \to \mathbb{R}$ is a function satisfying $g(x) = o(x^2)$ as $x \to 0$, and let*

$$S_{n,g} := \sum_{i=1}^{n} g(X_{n,i}).$$

Then for every $\varepsilon \in \mathbb{R} > 0$ we have that

$$\lim_{n\to\infty} \mathbb{P}(|S_{n;g}| \geq \varepsilon) = 0.$$

Proof. As in Lemma 37, we apply Chebyshev's inequality to obtain that

$$\mathbb{P}(|S_{n,g} - \mathbb{E}[S_{n,g}]| \geq \varepsilon) \leq \frac{\operatorname{Var}(S_{n,g})}{\varepsilon^2}. \tag{4}$$

From Lemmas 39 and 40 we have that $\lim_{n\to\infty} \mathbb{E}(S_{n;g}) = 0 = \lim_{n\to\infty} \operatorname{Var}(S_{n;g})$, so taking the limit as $n \to \infty$ in equation (4) gives the desired vanishing. □

From this theorem we can quickly deduce that quadratic approximations often suffice for understanding stochastic integrals of Brownian processes.

Corollary 42 (Quadratic approximation)**.** *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function which admits a quadratic Taylor approximation*

$$\widetilde{f}(x) := a + bx + cx^2 = f(x) + o(x^2)$$

as $x \to 0$, and let $X_{n,i} \stackrel{dist.}{=} \mathcal{N}(0, \frac{\sigma^2}{n})$ be i.i.d random variables. Then

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(X_{n,i}) = \lim_{n\to\infty} \sum_{i=1}^{n} \widetilde{f}(X_{n,i}).$$

Proof. By assumption we know that the function $g(x) := f(x) - \widetilde{f}(x) = o(x^2)$ as $x \to 0$ satisfies the conditions of Theorem 41. Therefore $\sum_{i=1}^{n} g(X_{n,i})$ converges in probability to zero as $n \to \infty$, proving the result. □

4.3 Recovering Itô calculus

With the quadratic approximation result in Corollary 42, to understand the probabilistic limit for general $g(x)$ which admit asymptotic series expansions around zero, we only need to compute it for the functions $g(x) = x^k$ when $k = 0$, 1 or 2.

Lemma 43 (Integrating constants)**.** *Suppose that $g(x) = a$ is constant function on $\mathbb{R}$ and that $\{X_{n,i}\}_{n\in\mathbb{N},1\leq i\leq n}$ is a family of i.i.d. random variables with $X_{n,i} \stackrel{dist.}{=} \mathcal{N}(0, \frac{\sigma^2}{n})$. Then the limit in probability*

$$\lim_{n\to\infty} \sum_{i=1}^{n} g(X_{n,i})$$

does not exist unless $a = 0$, in which case the limit is zero.

Proof. Since $g(X_{n,i}) = a$ this limit becomes

$$\lim_{n\to\infty} \sum_{i=1}^{n} g(X_{n,i}) = \lim_{n\to\infty} \sum_{i=1}^{n} a = \lim_{n\to\infty} n \cdot a$$

which diverges when $a \neq 0$ and is identically zero when $a = 0$. □

Lemma 44 (Integrating linear functions)**.** *Suppose that $g(x) = bx$ is a linear function on $\mathbb{R}$ and that $\{X_{n,i}\}_{n\in\mathbb{N},1\leq i\leq n}$ is a family of i.i.d. random variables with $X_{n,i} \overset{dist.}{=} \mathcal{N}(0, \frac{\sigma^2}{n})$. Then the limit in probability*

$$\lim_{n\to\infty}\sum_{i=1}^{n} g(X_{n,i}) \overset{dist.}{=} b\cdot\mathcal{N}(0,\sigma^2) \overset{dist.}{=} \mathcal{N}(0,b^2\sigma^2)$$

is a normal random variable.

Proof. By substituting $g(X_{n,i}) = b\cdot X_{n,i}$, this limit becomes

$$\lim_{n\to\infty}\sum_{i=1}^{n} g(X_{n,i}) = b\cdot\lim_{n\to\infty}\sum_{i=1}^{n} X_{n,i} = b\cdot\mathcal{N}(0,b^2\sigma^2).$$

□

Lemma 45 (Integrating quadratic functions)**.** *Suppose that $g(x) = cx^2$ is a quadratic function on $\mathbb{R}$ and that $\{X_{n,i}\}_{n\in\mathbb{N},1\leq i\leq n}$ is a family of i.i.d. random variables with $X_{n,i} \overset{dist.}{=} \mathcal{N}(0, \frac{\sigma^2}{n})$. Then the limit in probability*

$$\lim_{n\to\infty}\sum_{i=1}^{n} g(X_{n,i}) \overset{dist.}{=} c\sigma^2$$

is a constant random variable.

Proof. By substituting $g(X_{n,i}) = c(\cdot X_{n,i})^2$, this limit becomes

$$\lim_{n\to\infty}\sum_{i=1}^{n} g(X_{n,i}) = c\cdot\lim_{n\to\infty}\sum_{i=1}^{n}(X_{n,i})^2.$$

If we let $S_n := \sum_{i=1}^{n}(X_{n,i})^2$ then we see that it's expeceted value

$$\mathbb{E}[S_n] = \sum_{i=1}^{n}\mathbb{E}[(X_{n,i})^2] = \sum_{i=1}^{n}\mathrm{Var}[X_{n,i}] = \sum_{i=1}^{n}\tfrac{\sigma^2}{n} = \cancel{n}\cdot\tfrac{\sigma^2}{\cancel{n}} = \sigma^2.$$

and variance

$$\begin{aligned}\mathrm{Var}[S_n] &= \sum_{i=1}^{n}\mathrm{Var}[(X_{n,i})^2] = n\cdot\mathrm{Var}[(X_{n,i})^2] = n\cdot\left(\mathbb{E}[(X_{n,i})^4] - \mathbb{E}[(X_{n,i})^2]^2\right)\\ &= n\cdot\left[3\left(\frac{\sigma^2}{n}\right)^2 - \left(\frac{\sigma^2}{n}\right)^2\right] = \cancel{n}\cdot\frac{2\sigma^4}{n^{\cancel{2}}} = \frac{2\sigma^4}{n}.\end{aligned}$$

Therefore $S := \lim_{n\to\infty} S_n$ has $\mathbb{E}[S] = \sigma^2$ and $\mathrm{Var}[S] = 0$. Now applying Chebyshev's inequality

$$\mathbb{P}(|S - \mathbb{E}[S]| > \varepsilon) \leq \frac{\mathrm{Var}[S]}{\varepsilon^2}$$

to S and taking $\varepsilon > 0$ to be arbitrarily small shows that S is the constant random variable σ^2, proving the lemma. □

With these explicit computations and the quadratic approximation results of the previous section, we are now in a position to recover the rules of Itô calculus.

Theorem 46 (Recovering Itô calculus)**.** *Suppose that $g(x)$ is a function on $\mathbb{R}$ with $g(0) = 0$ that admits a quadratic Taylor approximation*

$$g(x) = bx + cx^2 + o(x^2)$$

near $x = 0$ and that $\{X_{n,i}\}_{n\in\mathbb{N},1\leq i\leq n}$ is a family of i.i.d. random variables with $X_{n,i} \overset{dist.}{=} \mathcal{N}(0, \frac{\sigma^2}{n})$. Then the limit in probability

$$\lim_{n\to\infty} \sum_{i=1}^{n} g(X_{n,i}) \overset{dist.}{=} b\mathcal{N}(0, \sigma^2) + c\sigma^2 \overset{dist.}{=} \mathcal{N}(c\sigma^2, b^2\sigma^2).$$

This agrees with the Itô's Lemma when $g(x, t) = g(x)$ is a C^2-function independent of time, which states that the stochastic integral

$$\int_{t=0}^{T} g(W_t)\, dW_t = g'(0)W_T + \frac{g''(0) \cdot T}{2}$$

where $\{W_t\}_{t\geq 0}$ is the standard Brownian motion process and when $T = \sigma^2$.

Proof. The first assertion follows from combining Corollary 42 and Lemmas 43 – 45. The agreement with Itô calculus follows since in the Taylor approximation we know that $b = g'(0)$ and $c = \frac{g''(0)}{2}$. □

4.4 Approximately constant-weighted portfolios

We now extend the results of Section 3.2 to describe the log-return random variables for an instantaneously rebalanced constant-weighted portfolio in the antisymmetric two stock market $\mathcal{M}$. Since we have shown that the probabilistic approach recovers the rules of Itô calculus, the log-value process we obtain is the same as the log-value process given by Stochastic Portfolio Theory.

Lemma 47 (Buy-and-hold log-returns)**.** *For any $w_1, w_2 \in \mathbb{R}_{\geq 0}$ we have that*

$$\mathrm{LogRet}(\mathrm{BH}_{(w_1,w_2)}, [T_0, T_1]) = \ln\left(w_1 e^r + w_2 e^{-r}\right)$$

where $r = \mathcal{N}(0, T_1 - T_0)$.

Proof. This follows directly from the definition of the buy-and-hold final portfolio. □

The following picture with $\vec{w} = (w_1, w_2) = (\frac{1}{3}, \frac{2}{3})$ is useful for visualizing some of the important differences between the buy-and-hold portfolios whose weights are equal (i.e. $w_1 = w_2$), and the more general case where $w_1 \neq w_2$.

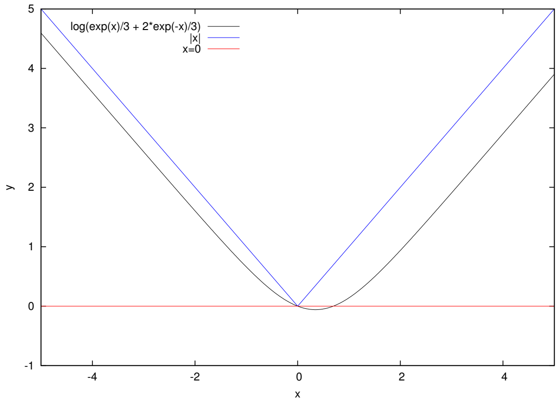

Figure 3: Plots of $\ln(\frac{1}{3}e^x + \frac{2}{3}e^{-x})$ and $|x|$ as functions of x.

Remark 48 (Failure of LLN bounds)**.** *To proceed as we did for constant-weighted portfolios, we would need to establish a bound of the form*

$$0 \leq \ln\left(w_1 e^x + w_2 e^{-x}\right) \leq |x|^\alpha$$

for some $\alpha > 1$ when $w_1 \neq w_2$ and $w_1 + w_2 = 1$. While this is true when $\alpha = 1$ (as illustrated above), for $\alpha > 1$ the tangent line to $|x|^\alpha$ at $x = 0$ is horizontal, and this forbids the existence of such an upper bound. This is an essential point, since it allows for a non-deterministic random variable to appear as the limit.

With the failure of a LLN bound, the most we can hope to show along these lines is how the expected value of the log-return process will behave.

Theorem 49 (Linearity of expected value)**.** *For any $w_1, w_2 \in \mathbb{R}_{\geq 0}$ with $w_1 + w_2 = 1$,*

$$\mathbb{E}[\mathrm{LogRet}(\mathrm{CW}_{(w_1,w_2)}, [T_0, T_1])] = \kappa(T_1 - T_0)$$

where

$$\kappa := \frac{d}{dt^+}\left(\mathbb{E}\left[\mathrm{LogRet}(\mathrm{BH}_{(w_1,w_2)}, [0,t])\right]\right|_{t=0}.$$

Proof. From the argument in Lemma 18 we have that

$$\mathrm{LogRet}(\mathrm{CW}_{(w_1,w_2)}, [T_0, T_1]) = \mathrm{LogRet}(\mathrm{CW}_{(w_1,w_2)}, [0, 1]).$$

Then applying the scaling argument in Theorem 17 gives that

$$\mathbb{E}[\mathrm{LogRet}(\mathrm{CW}_{(w_1,w_2)}, [0,1])] = \frac{d}{dt^+}\left(\mathbb{E}\left[\mathrm{LogRet}(\mathrm{BH}_{(w_1,w_2)}, [0,t])\right]\right|_{t=0}$$

which proves the theorem. $\square$

Theorem 50 (Computing the expected value slope)**.**

$$\frac{d}{dt^+}\left(\mathbb{E}\left[\mathrm{LogRet}(\mathrm{BH}_{(w_1,w_2)},[0,t])\right]\right|_{t=0} = \frac{1-(w_1-w_2)^2}{2}.$$

Proof. By applying the initial substitutions and justifications from the L'Hopital's rule argument in Remark 23, we see that

$$\begin{aligned}\frac{d}{dt^+}\left(\mathbb{E}\left[\mathrm{LogRet}(\mathrm{BH}_{(w_1,w_2)},[0,t])\right]\right|_{t=0} &= \lim_{\sigma\to 0^+}\frac{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(y\sigma)e^{\frac{-y^2}{2}}\,dy}{\sigma^2}\\ &\overset{(2\times\text{ L'Hopital})}{=} \lim_{\sigma\to 0^+}\frac{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f''(y\sigma)y^2e^{\frac{-y^2}{2}}\,dy}{2},\end{aligned}$$

where $f(x) := \ln(w_1e^x + w_2e^{-x})$. To compute $f''(x)$ we let

$$h_+(x) := w_1e^x + w_2e^{-x}, \qquad h_-(x) := \tfrac{dh_+(x)}{dx} = w_1e^x - w_2e^{-x}$$

and see that

$$f'(x) = \frac{h_-(x)}{h_+(x)}, \qquad f''(x) = \frac{(h_+(x))^2-(h_-(x))^2}{(h_+(x))^2} = 1-\left(\frac{h_-(x)}{h_+(x)}\right)^2,$$

and so as $\sigma\to 0^+$ we see that $f''(\sigma y)$ approaches $1-\left(\frac{w_1-w_2}{w_1+w_2}\right)^2 = 1-(w_1-w_2)^2$ uniformly on compact subsets. Applying this to the previous limit we see that the desired right-derivative is given by $\frac{1-(w_1-w_2)^2}{2}$. □

Corollary 51 (Expected log-returns)**.** *For $w_1, w_2 \in \mathbb{R}_{\geq 0}$ with $w_1 + w_2 = 1$, the expected log-returns for a constant-weighted portfolio is given by*

$$\mathbb{E}\left[\mathrm{LogRet}(\mathrm{CW}_{(w_1,w_2)},[T_0,T_1])\right] = \tfrac{1-(w_1-w_2)^2}{2}(T_1-T_0)$$

almost surely.

Proof. This follows directly from combining Theorems 49 and 50. □

While the previous lemmas allow us to understand the expected value of the log-return process, the failure of a LLN bound means that we cannot use these ideas to determine the distribution of the limiting log-return process. To do this, we appeal to the results of the previous section and the direct evaluation of the limiting random variable via its quadratic approximation.

Lemma 52 (Computing derivatives)**.** *For given $w_1, w_2 \in \mathbb{R}_{\geq 0}$ with $w_1 + w_2 = 1$, let $g(r) :=$ $\mathrm{LogRet}(\mathrm{BH}_{(w_1,w_2)},[T_0,T_1])$. Then we have the derivatives*

$$\begin{aligned} g'(0) &= \left.\frac{dg}{dr}\right|_{r=0} = w_1 - w_2,\\ g''(0) &= \left.\frac{d^2g}{dr^2}\right|_{r=0} = 1-(w_1-w_2)^2.\end{aligned}$$

Proof. From the explicit formula in Lemma 47, we have that

$$g'(r) = \frac{w_1 e^r - w_2 e^{-r}}{w_1 e^r + w_2 e^{-r}}$$

and by the quotient rule

$$\begin{aligned} g''(r) &= \frac{(w_1 e^r + w_2 e^{-r})(w_1 e^r + w_2 e^{-r}) - (w_1 e^r - w_2 e^{-r})(w_1 e^r + w_2 e^{-r})}{(w_1 e^r + w_2 e^{-r})^2} \\ &= \frac{(w_1 e^r + w_2 e^{-r})^2 - (w_1 e^r - w_2 e^{-r})^2}{(w_1 e^r + w_2 e^{-r})^2}. \end{aligned}$$

Now substituting $r = 0$ and using the relation that $w_1 + w_2 = 1$, we have that

$$g'(0) = \frac{w_1 - w_2}{\cancel{w_1 + w_2}} = w_1 - w_2$$

and

$$g''(0) = \frac{1 - (w_1 - w_2)^2}{\cancel{(w_1 + w_2)^2}} = 1 - (w_1 - w_2)^2,$$

which proves the lemma. □

Theorem 53 (Constant-weighted log-returns)**.** *The log-return of the constant-weighted portfolio strategy* $\mathrm{CW}_{(w_1,w_2)}$ *where* $w_1 + w_2 = 0$ *in the anti-symmetric market* $\mathcal{M}$ *is given by the random variable*

$$\begin{aligned} \mathrm{LogRet}(\mathrm{CW}_{(w_1,w_2)}, [T_0, T_1]) &\overset{dist.}{=} \tfrac{(1-(w_1-w_2)^2)(T_1-T_0)}{2} + (w_1 - w_2) \cdot \mathcal{N}(0, T_1 - T_0) \\ &\overset{dist.}{=} \mathcal{N}(\tfrac{T_1-T_0}{2} + \tfrac{(w_1-w_2)^2(T_1-T_0)}{2}, (w_1 - w_2)^2(T_1 - T_0)). \end{aligned}$$

Proof. By definition we have that

$$\mathrm{LogRet}(\mathrm{CW}_{(w_1,w_2)}, [T_0, T_1]) = \lim_{\substack{n\to\infty \\ prob.}} \sum_{i=0}^{n-1} \mathrm{LogRet}(\mathrm{BH}_{(w_1,w_2)}, [t_i, t_{i+1}])$$

and by Theorem 46 we see that this limit converges in probability to the random variable $b\mathcal{N}(0, \sigma^2) + c\sigma^2$ where $br + cr^2$ is the quadratic Taylor approximation to $g(r) = \mathrm{LogRet}(\mathrm{BH}_{(w_1,w_2)}, [T_0, T_1])$. By Lemmas 47 and 52, we see that $b = g'(0) = w_1 - w_2$ and $c = \frac{g''(0)}{2} = \frac{1-(w_1-w_2)^2}{2}$, which proves the theorem. □

Remark 54 (Agreement with SPT log-value process)**.** *Notice that the log-value processes in Theorem 36 given by Stochastic Portfolio Theory and in 53 by taking a probabilistic limit agree. This justifies the use of stochastic Itô differentials to understand a limit of buy-and-hold portfolios, and shows that as the time partition becomes finer that the associated log-value process for the approximately constant-weighted portfolio strategy will approach (in probability) the log-value process of the instantaneously rebalanced constant-weighted strategy.*

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