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AN EXACT MASS FORMULA FOR QUADRATIC FORMS OVER NUMBER FIELDS

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ABSTRACT

In this paper we give an explicit formula for the mass of a quadratic form in $n \geq 3$ variables with respect to a maximal lattice over an arbitrary number field k. We make the technical assumption that the determinant of the form is a unit up to a square if n is odd. The corresponding formula for k totally real was recently computed by Shimura [Shi].

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CHAPTER 0

SUMMARY

Our goal is to give an exact formula for the mass of the genus of a quadratic form φ on a maximal lattice defined over an arbitrary number field k. In Section 2 we explain how knowledge of the Tamagawa number of the special orthogonal group G^φ gives rise to a mass formula. Such a formula expresses the mass as a product of local factors over all places v of k, so our problem is reduced to computing each of these. For the non-archimedian places, these factors were recently computed by Shimura [Shi]. We state his result in Section 3 and for completeness include a translation between our language and his. In Section 4 we compute the archimedian factors, treating separately the 3 cases: v real, φ definite; v real, φ indefinite; and v complex. To define the factors in the last two cases, we choose a symmetric space $\mathfrak{Z}_v$ equipped with a G^φ_v action and a non-zero G^φ_v invariant volume form $\omega_{\mathfrak{Z}}$. Finally, in Section 5 we compute the mass of φ with respect to a maximal lattice. We note that this formula agrees with Shimura's in the case of k totally real. Our results depend on several technical lemmas which we include in the Appendix.

CHAPTER 1

INTRODUCTION

We begin with a quadratic space (V,φ) over an algebraic number field k. By this we mean a k vector space V together with a non-degenerate quadratic form $\varphi : V \longrightarrow k$. Let O_k denote the ring of integers of k and let O_v denote the local ring of integers at each place v of k. We consider (V,φ) as well as its localizations (V_v,φ_v) given by linear extension of scalars to k_v. Given a lattice $\Lambda \subset (V,\varphi)$, we have the associated local lattice $\Lambda_{\mathfrak{p}} = \Lambda \otimes_{O_k} O_{\mathfrak{p}} \subset (V_{\mathfrak{p}},\varphi_{\mathfrak{p}})$ at each non-archimedian place $\mathfrak{p}$ of k. We write (Λ,φ) for the restriction of the form (V,φ) to Λ, and $(\Lambda_{\mathfrak{p}},\varphi_{\mathfrak{p}})$ for the restriction of $(V_{\mathfrak{p}},\varphi_{\mathfrak{p}})$ to $\Lambda_{\mathfrak{p}}$.

With (V,φ) as above, we let $G^{\varphi} = G(\varphi)$ be the special orthogonal group of (V,φ) by which we mean the group of deteminant 1 invertible linear transformations of V which preserve φ. We also define G_v^{φ} to be the special orthogonal group of (V_v,φ_v). Then we have a natural G^{φ} action on (V,φ), and a natural G_v^{φ} action on (V_v,φ_v). We say that two lattices $\Lambda,\Lambda' \subseteq (V,\varphi)$ are **globally equivalent** if there exists $g \in G^{\varphi}$ such that $\Lambda' = g\Lambda$, and **locally equivalent** if for each $v \in \mathbf{h}$, there exists $g_v \in G_v^{\varphi}$ such that $\Lambda'_v = g_v\Lambda_v$. We define the **genus** of (Λ,φ) to be the set of all lattices locally equivalent to (Λ,φ), and say that the **classes** of (Λ,φ) are the global equivalence classes of (Λ,φ) in its genus.

Let $G_{\mathbf{A}}^{\varphi}$ be the adelization of G^{φ}. Then there is a natural $G_{\mathbf{A}}^{\varphi}$ action on the space of lattices $\Lambda \subseteq (V,\varphi)$. To see this, take $g = (g_v) \in G_{\mathbf{A}}^{\varphi}$ and define $g\Lambda$ to be the lattice $\Lambda'' \subseteq (V,\varphi)$ such that $\Lambda''_v = g_v\Lambda_v$ for all $v \in \mathbf{h}$.

Let $\mathfrak{Cl}$ denote the (finite) set of classes in the genus of (Λ, φ), and take $\{\Lambda^a\}_{a \in \mathfrak{Cl}}$ to be a complete set of representative lattices in (V, φ) for the classes of Λ. We denote by Γ^a the group of **automorphisms** of (Λ^a, φ), defined to be those $g \in G^\varphi$ leaving Λ^a invariant. If we are working with a totally definite lattice (Λ, φ) over a totally real number field, we define the mass of its genus to be

$$\mathrm{Mass}(\Lambda, \varphi) = \sum_{a \in \mathfrak{Cl}} [\Gamma^a : 1]^{-1},$$

For an arbitrary lattice (Λ, φ), $[\Gamma^a : 1]$ is not necessarily finite, but we would still like to keep track of the size of Γ^a. To do this we let Γ^a act on some symmetric space $\mathfrak{Z}$ and choose a measure on $\mathfrak{Z}$ invariant under this action. We then define the mass in terms of the measures of the quotients $\Gamma^a \backslash \mathfrak{Z}$. So in general we define the **mass** of (Λ, φ) to be

$$\mathrm{Mass}(\Lambda, \varphi) = \sum_{a \in \mathfrak{Cl}} \nu(\Gamma^a), \tag{1.1}$$

where

$$\nu(\Gamma^a) = \begin{cases} [\Gamma^a : 1]^{-1} & \text{if } G_{\mathbf{a}} \text{ is compact,} \\ [\Gamma^a \cap \{\pm 1\} : 1]^{-1} \mathrm{vol}(\Gamma^a \backslash \mathfrak{Z}) & \text{otherwise.} \end{cases}$$

In the case where (Λ, φ) is a maximal lattice for (V, φ) (i.e., maximal for the property $\varphi(\Lambda) \subseteq O_k$), we will give an exact formula for $\mathrm{Mass}(\Lambda, \varphi)$. This formula essentially expresses the mass as a product of even integer values of the Dedekind zeta function of k, a power of the index of Λ in its dual lattice, and some gamma function factors. If $2 | \dim_k(V)$ a special value of the L-function of a certain quadratic extension of k also appears.

CHAPTER 1.1

SUMMARY OF NOTATION

Throughout this paper we take k to be a number field, O_k its ring of integers, and D_k the discriminant of $k/\mathbb{Q}$. We denote by v a valuation (or place) of k. We also let $\mathbf{a}$ and $\mathbf{h}$ denote the archimedian and non-archimedian places of k respectively. Suppose $\mathfrak{p}$ is a prime ideal in O_k lying over the prime p in $\mathbb{Z}$, and $x \in k$. We let $|x|_{\mathfrak{p}}$ denote the usual $\mathfrak{p}$-adic absolute value of x defined by $|x|_{\mathfrak{p}} = q^{-\mathrm{ord}_{\mathfrak{p}}(x)}$, where we take $q = q_{\mathfrak{p}} = [O_{\mathfrak{p}} : \mathfrak{p}]$.

We follow the convention that if we have an object R defined at a certain valuation v, we denote it by R_v. If R_v is defined at each of the archimedian valuations, we also write

$$R_{\mathbf{a}} = \prod_{v \in \mathbf{a}} R_v.$$

For an algebraic group G defined over k, we denote the adelization of G by $G_{\mathbf{A}}$.

If R is an arbitrary set, we denote by R_n^m the $m \times n$ matrices with coefficients in R. We write the transpose of a matrix A as ${}^tA$. If x is a matrix, then we let x_{ij} denote the entry of x in the i^{th} row and j^{th} column. Conversely given numbers x_{ij}, we let (x_{ij}) denote the matrix whose entries satisfy $(x_{ij})_{ij} = x_{ij}$. We abbreviate the diagonal matrix

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$$

by $\mathrm{diag}[a_{11}, \cdots, a_{nn}]$, and denote the $n \times n$ identity matrix by 1_n. Given an arbitrary $n \times n$ matrix A and an integer l with $1 \leq l \leq n$, we define $\det_l(A)$ to

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be the determinant of the upper left $l \times l$ submatrix of A. If A is a matrix of functions, we define the matrix of 1-forms $dA = (dA_{ij})$. Given two $n \times n$ matrices A and B over $\mathbb{R}$, we say that $A > B$ if the matrix $A - B$ is positive definite, and we set

$$S_+^n = \{A \in \mathbb{R}_n^n \mid {}^tA = A > 0\}.$$

We let (V, φ) denote a non-degenerate quadratic space of dimension n over k, and take $V_v, \Lambda_\mathfrak{p}, G^\varphi, G_v^\varphi, G_\mathbf{A}^\varphi$ as defined in the introduction. If we choose a basis $\{v_1, \cdots, v_n\}$ for V, we may express φ as the matrix $\psi = [\varphi(v_i, v_j)]_{1 \leq i,j \leq n}$. We also let $G^-(\varphi)$ denote the set of invertible linear transformations of V which preserve the form φ and have determinant -1.

For convenience, we define the symbols

$$T = \text{ \{ Symmetric } n \times n \text{ matrices with coefficients in } k\} \text{ ,}$$

$$X = k_n^n,$$

and their local counterparts T_v, and X_v at a valuation v by replacing k by k_v in the above definition. We also let S^{n-1} denote the unit sphere in $\mathbb{R}^n$.

We set $i = \sqrt{-1} \in \mathbb{C}$. For $x \in \mathbb{R}$ we let $\lfloor x \rfloor$ be the greatest integer $\leq x$. Also, when there is no danger of confusion, we freely use the letters i, j, k, l as indices. Our equations and statements are numbered first by section, then by order within each section, with the appendix labeled by A (e.g. Lemma A2).

CHAPTER 2

THE TAMAGAWA NUMBER AND LOCAL FACTORS

The main fact that we use in our result is that the Tamagawa number τ of the special orthogonal group $G = G^{\varphi}$ over any number field k is

$$\tau(G) = 2 \qquad \text{if } n \geq 3, \tag{2.1}$$

where $n = \dim_k(V)$. To define the Tamagawa number we first choose a measure $(dx)_{\mathbf{A}}$ on $k_{\mathbf{A}}$ normalized so that

$$\int_{k \backslash k_{\mathbf{A}}} (dx)_{\mathbf{A}} = 1. \tag{2.2}$$

We then define the **Tamagawa number** of G to be

$$\tau(G) = \int_{G \backslash G_{\mathbf{A}}} |\omega_G|_{\mathbf{A}}, \tag{2.3}$$

where ω_G is a non-zero left G invariant top degree differential form on G and $|\omega_G|_{\mathbf{A}}$ is the volume element defined with respect to $(dx)_{\mathbf{A}}$. By the product formula we see $|c\omega_G|_{\mathbf{A}} = |\omega_G|_{\mathbf{A}}$ for $c \in k^{\times}$, and since ω_G is chosen from a 1 dimensional space, this specifies a left G invariant measure on $G_{\mathbf{A}}$ independently of our choice of ω_G. We call the measure associated to ω_G the **Tamagawa measure** on $G_{\mathbf{A}}$. (For a more detailed introduction, see [Tam], [Vos], or [Weil].)

From now on when speaking of an invariant object, we always understand this to mean it is left invariant. For clarity we also define a **volume form** to be a nowhere zero diferential form of top degree.

In our computations, we define another measure $(d'x)_{\mathbf{A}}$ by the restricted product $(d'x)_{\mathbf{A}} = \prod'_v (d'x)_v$ with local measures

$$(d'x)_v = \begin{cases} \text{Haar measure on } k_v \text{ normalized by } \int_{O_{\mathfrak{p}}} (d'x)_v = 1 & \text{if } k_v = k_{\mathfrak{p}}, \\ \text{Lesbegue measure on } \mathbb{R} & \text{if } k_v = \mathbb{R}, \\ idz \wedge d\bar{z} = 2 \times \text{Lesbegue measure on } \mathbb{R}^2 & \text{if } k_v = \mathbb{C}. \end{cases}$$

Then we have $\int_{k\backslash k_{\mathbf{A}}} (d'x)_{\mathbf{A}} = |D_k|^{1/2}$. So in terms of $(d'x)_{\mathbf{A}}$ we have

$$\begin{aligned} \tau(G) &= |D_k|^{\frac{-\dim_k(G)}{2}} \int_{G\backslash G_{\mathbf{A}}} |\omega_G|'_{\mathbf{A}} \\ &= |D_k|^{\frac{-n(n-1)}{4}} \int_{G\backslash G_{\mathbf{A}}} |\omega_G|'_{\mathbf{A}}. \end{aligned} \tag{2.4}$$

Here $|\omega_G|'_{\mathbf{A}}$ is the volume element derived from ω_G using $(d'x)_{\mathbf{A}}$ instead of $(dx)_{\mathbf{A}}$.

We now construct a suitable volume form ω_G on G^φ. Choose a basis $\{v_1, \cdots, v_n\}$ for (V, φ) and use it to write φ as a matrix ψ. This gives a natural map

$$\begin{aligned} X = (k)^n_n &\xrightarrow{\mathcal{F}} T \\ x &\longmapsto {}^t x \psi x, \end{aligned} \tag{2.5}$$

whose fibre over the matrix $\psi \in T$ is the full orthogonal group of φ. Given the non-zero volume forms

$$\omega_X = \bigwedge_{i,j} dx_{ij}, \qquad \omega_T = \bigwedge_{i \le j} dt_{ij} \tag{2.6}$$

on X and T respectively, we can find a form ω on X such that

$$\omega_X = \mathcal{F}^*(\omega_T) \wedge \omega. \tag{2.7}$$

Pulling ω back to the fibre and then restricting to the identity component we get a form ω_G on G^φ. By Lemma A6, ω_G is a non-zero G^φ invariant volume form, independent of our choice of ω. We will use this construction many times in our calculation.

For each $v \in \mathbf{a} \cup \mathbf{h}$ we define

$$\beta_v(\psi) = \beta_v(\Lambda, \psi) = \frac{1}{2} \lim_{U \to \psi_v} \frac{\int_{U'} dX}{\int_U dT}, \tag{2.8}$$

where $dX = \prod_{i,j} (dx_{ij})_v$ and $dT = \prod_{i \le j} (dt_{ij})_v$ are the measures associated to ω_X and ω_T in these coordinates,

$$U' = \begin{cases} \mathcal{F}^{-1}(U) & \text{if } v \in \mathbf{a}, \\ \mathcal{F}^{-1}(U) \cap \{x \in X_v \mid x\Lambda_v = \Lambda_v\} & \text{if } v \in \mathbf{h}, \end{cases}$$

and U is an open neighborhood of ψ_v in T_v. One should note that $\beta_v(\psi)$ depends not only on (V, φ) and v, but also on our given choice of basis for (V, φ). In our calculations the lattice Λ will be fixed, so we will often supress Λ and write $\beta_v(\psi)$.

We define $G_{\mathbf{a}}$ to be the product of the archimedean localizations of G and use a particular choice of volume form ω_G in (2.3) to define an archimedean measure $\tau_{\mathbf{a}}$ on it using $\prod_{v \in \mathbf{a}} |\omega_G|'_v$. By writing (2.1) in terms of its local measures one can prove the following result:

Theorem 2.1. *Let Λ be a lattice in (V, φ), and ψ a matrix representing φ in some basis. Then*

$$\sum_{a \in \mathfrak{Cl}} \tau_{\mathbf{a}}(\Gamma^a \backslash G^{\varphi}_{\mathbf{a}}) = \tau(G^{\varphi}) \prod_{v \in \mathbf{h}} \beta_v(\Lambda, \psi)^{-1},$$

with $\tau_{\mathbf{a}}$ and $\beta_v(\Lambda, \psi)$ as above, and Γ^a defined in §1.

Proof. This is proved in [Cas, pp380-382] when $k = \mathbb{Q}$, but the argument there works for any number field k. In his notation $\beta_v(\Lambda, \psi) = \lambda_v = \tau_v(O^+(\Lambda_v))$ and the right side of [Cas, Appendix B (4.19), p382] should read $2\lambda_\infty^{-1} \prod_{p \ne \infty} \lambda_p^{-1}$. $\square$

To simplify our calculations, we change basis locally so that ψ_v has the standard form

$$\phi_v = {}^t\sigma_v \psi_v \sigma_v = \begin{cases} \begin{bmatrix} 0 & 0 & 2^{-1}1_r \\ 0 & \theta_v & 0 \\ 2^{-1}1_r & 0 & 0 \end{bmatrix} & \text{if } k_v = k_{\mathfrak{p}}, \\ \begin{bmatrix} 1_q & 0 \\ 0 & -1_r \end{bmatrix} & \text{if } k_v = \mathbb{R}, \\ 1_n & \text{if } k_v = \mathbb{C}, \end{cases} \tag{2.9}$$

for some invertible matrix $\sigma_v \in (k_v)^n_n$, where $q, r \in \mathbb{N}$ satisfying either $q + r = n$ and $q \geq r$ or $\dim(\theta_v) + 2r = n$, and θ_v is some anisotropic symmetric matrix with $\dim(\theta_v) \leq 4$.

Further, if we take Λ to be a maximal lattice, by [Shi2, Lemma 5.6], locally we can choose a free $O_\mathfrak{p}$-basis for $\Lambda_\mathfrak{p}$ so that $(\Lambda_\mathfrak{p}, \varphi_\mathfrak{p})$ is represented by $\phi_\mathfrak{p}$ above. We choose the matrices $\sigma_\mathfrak{p}$ so this is true.

CHAPTER 3

NON-ARCHIMEDIAN LOCAL FACTORS

The non-archmedian local factors that appear in the mass formula for a maximal lattice Λ have been calculated by Shimura in [Shi], under the condition that the determinant of φ is a unit up to a square if n is odd. We now show how the local factors in [Shi] relate to the local factors $\beta_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}, \phi_{\mathfrak{p}})$ in our mass formula.

Fix a basis for $V_{\mathfrak{p}}$, let ψ be the invertible $n \times n$ matrix defined over $k_{\mathfrak{p}}$ which represents $(V_{\mathfrak{p}}, \varphi_{\mathfrak{p}})$ in this basis, and let $\Lambda_{\mathfrak{p}}$ be a lattice in $(V_{\mathfrak{p}}, \varphi_{\mathfrak{p}})$ (i.e., $\Lambda_{\mathfrak{p}}$ is a compact $O_{\mathfrak{p}}$-module such that $\Lambda_{\mathfrak{p}} \otimes_{O_k} k_{\mathfrak{p}} = V_{\mathfrak{p}}$). We define $\beta_{\mathfrak{p}}(\psi)$ as in §2 to be the limit of the ratio of volumes

$$\beta_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}, \psi) = \frac{1}{2} \lim_{U \to \psi} \frac{\int_{U'} dX}{\int_U dT}, \tag{3.1}$$

where U' is a neighborhood in $X_{\mathfrak{p}}$ determined by $\Lambda_{\mathfrak{p}}$ and an open neighborhood U of ψ in $T_{\mathfrak{p}}$. We may also write U' as $U'(\psi)$ to emphasize its dependence on the matrix ψ. Since we are working over a $\mathfrak{p}$-adic field, we have a natural choice of neighborhoods U_i to use for this limit, namely $U_i = \psi + P_i$ where $P_i = (\mathfrak{p}^i)_n^n \cap T_{\mathfrak{p}}$.

Lemma 3.1. *Let $\Lambda_{\mathfrak{p}}, \psi$ be as above and let $c \in k_{\mathfrak{p}}^{\times}$. Then we have*

$$\beta_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}, \psi) = |c|_{\mathfrak{p}}^{\frac{n(n+1)}{2}} \beta_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}, c\psi) = |\det(c \cdot 1_n)|_{\mathfrak{p}}^{\frac{(n+1)}{2}} \beta_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}, c\psi).$$

Proof. We take our limit for $\beta_{\mathfrak{p}}$ with respect to the neighborhoods U_i. Consider the set

$$U_i'(\psi) = \{x \in X_{\mathfrak{p}} \mid {}^t x \psi x \in \psi + P_i \text{ and } x\Lambda_{\mathfrak{p}} = \Lambda_{\mathfrak{p}}\},$$

and notice $U_i'(\psi) = U_{i+\mathrm{ord}_\mathfrak{p}(c)}'(c\psi)$. From this we have

$$\begin{aligned}
\beta_\mathfrak{p}(\psi) &= \frac{1}{2} \lim_{i\to\infty} \frac{\int_{U_i'(\psi)} dX}{\int_{U_i} dT} \\
&= \frac{1}{2} \lim_{i\to\infty} \frac{\int_{U_{i+\mathrm{ord}_\mathfrak{p}(c)}'(c\psi)} dX}{\int_{U_i} dT} \\
&= |c|_\mathfrak{p}^{\frac{n(n+1)}{2}} \frac{1}{2} \lim_{i\to\infty} \frac{\int_{U_{i+\mathrm{ord}_\mathfrak{p}(c)}'(c\psi)} dX}{\int_{U_{i+\mathrm{ord}_\mathfrak{p}(c)}} dT} \\
&= |c|_\mathfrak{p}^{\frac{n(n+1)}{2}} \beta_\mathfrak{p}(c\psi),
\end{aligned}$$

which completes the proof. □

The following lemma relates our local factors to those in [Shi].

LEMMA 3.2. *Let $v \in \mathbf{a} \cup \mathbf{h}$ and suppose that $\psi' = {}^tA\psi A$ for some invertible $n \times n$ matrix A. Then we have*

$$\beta_v(\psi') = |\det(A)|_v^{n+1}\, \beta_v(\psi).$$

PROOF. Let $L_A : X \longrightarrow X$ denote left multiplication by the matrix A and define $[A] : T \longrightarrow T$ by $[A](t) = {}^tAtA$, which correspond to change of basis by A for a quadratic form.

Fix an open set U about ψ' in T, and let $V = [A]^{-1}(U)$ be the coresponding neighborhood of ψ. Then

$$\frac{\mathrm{vol}_X(\mathcal{F}_{\psi'}^{-1}(U))}{\mathrm{vol}_T(U)} \cdot \frac{\mathrm{vol}_T(U)}{\mathrm{vol}_T([A]^{-1}(U))} = \frac{\mathrm{vol}_X(\mathcal{F}_{\psi}^{-1}(V))}{\mathrm{vol}_T(V)}$$

since $\mathcal{F}_{\psi'} = [A] \circ \mathcal{F}_\psi$.

Since $\Lambda_\mathfrak{p}$ is an abstract lattice, it does not change under change of basis, so passing to the limit as $U \to \psi'$ we have

$$\beta_v(\psi') = \lim_{U\to\psi} \frac{\mathrm{vol}_T([A]^{-1}(U))}{\mathrm{vol}_T(U)} \beta_v(\psi).$$

This ratio of volumes is given by computing the pull-back of the volume form ω_T under the map $[A]$. We claim that

$$[A]^*(\omega_T) = \det(A)^{n+1}\omega_T$$

which is to say

$$\bigwedge_{i\leq j} d({}^tAtA)_{ij} = \det(A)^{n+1} \bigwedge_{i\leq j} dt_{ij}. \tag{3.2}$$

To see this notice that we already know (3.2) if we replace $\det(A)^{n+1}$ by some character $c(A)$ on $GL_n(k_v)$, since $[AB] = [B][A]$. By construction $c(A)$ is a polynomial in the entries of A. Since the only continuous characters on GL_n are powers of the determinant, we easily verify (3.2) by checking against the scalar matrices $A = \lambda \cdot 1_n$.

With this we have

$$\lim_{U\to\psi} \frac{\mathrm{vol}_T([A]^{-1}(U))}{\mathrm{vol}_T(U)} = |\det(A)|_v^{n+1},$$

which proves our lemma. $\square$

LEMMA 3.3. *Suppose we have a lattice $\Lambda_\mathfrak{p} \subset (V_\mathfrak{p}, \varphi_\mathfrak{p})$ and we choose a basis $\{v_1, \cdots, v_n\}$ for $V_\mathfrak{p}$ such that $\Lambda_\mathfrak{p} = \sum_{i=1}^n O_\mathfrak{p} v_i$. If ψ is the matrix representing $(V_\mathfrak{p}, \varphi_\mathfrak{p})$ in this basis and $\psi \in (O_{frakp})_n^n$, then $\beta_\mathfrak{p}(\Lambda_\mathfrak{p}, \psi) = \frac{1}{2} e_\mathfrak{p}(\psi)$, where $e_\mathfrak{p}(\psi)$ is as in [Shi, §8].*

PROOF. In [Shi,§8] $e_\mathfrak{p}(\psi)$ is defined in terms of points in $(O_\mathfrak{p}/\mathfrak{p}O_\mathfrak{p})_n^n$, so we need to show that the measures of U_i and U_i' can be found by counting the points of their respective images over the residue field. Since we have chosen our U_i to be a translate of P_i, this is true for U_i. We now show that U_i' is a (disjoint) union of translates of $P_i' = (\mathfrak{p}_i)_n^n$.

Let $x \in X_\mathfrak{p}$. From $\Lambda_\mathfrak{p} = \sum_{i=1}^n O_\mathfrak{p} v_i$ we see $x\Lambda_\mathfrak{p} \subseteq \Lambda_\mathfrak{p} \Leftrightarrow x \in (O_\mathfrak{p})_n^n$, and such an x fixes $\Lambda_\mathfrak{p}$ if in addition $|\det(x)|_\mathfrak{p} = 1$. Now consider $x + m$ with $x \in U_i'$ and

$m \in P'_i$. Expanding $\det(x+m)$ and applying the ultrametric inequality, we see $|\det(x+m)|_{\mathfrak{p}} = |\det(x)|_{\mathfrak{p}} = 1$ so $(x+m)\Lambda_{\mathfrak{p}} = \Lambda_{\mathfrak{p}}$. Also ${}^t(x+m)\psi(x+m) = \psi + m'$ with $m' \in P_i$, hence $x+m$ is in U'_i. Thus $x + P'_i \subseteq U'_i$, so U'_i is a union of translates of P'_i.

With this, we can compute the measures of U_i and U'_i by knowing the images of their components in the quotient $O_{\mathfrak{p}}/\mathfrak{p}^i O_{\mathfrak{p}}$. If $q = \#(O_{\mathfrak{p}}/\mathfrak{p}O_{\mathfrak{p}})$ and N'_i is defined to be the number of solutions x of ${}^t x\psi x \equiv \psi \bmod P'_i$, we have

$$\frac{\int_{U'_i} dX}{\int_{U_i} dT} = \frac{\left(\frac{1}{q^i}\right)^{n^2} N'_i}{\left(\frac{1}{q^i}\right)^{\frac{n(n+1)}{2}}} = q^{\frac{-n(n-1)}{2}i} N'_i.$$

Therefore

$$\beta_{\mathfrak{p}}(\psi) = \frac{1}{2}\lim_{U \to \psi} \frac{\int_{U'} dX}{\int_U dT} = \frac{1}{2}\lim_{i\to\infty} \frac{\int_{U'_i} dX}{\int_{U_i} dT} = \frac{1}{2}\lim_{i\to\infty} q^{\frac{-n(n-1)}{2}i} N'_i,$$

where the last equality is by definition the number $\frac{1}{2}e_{\mathfrak{p}}(\psi)$ in [Shi,§8]. $\square$

Take $\Lambda_{\mathfrak{p}}$ to be a maximal lattice in $(V_{\mathfrak{p}}, \varphi_{\mathfrak{p}})$, and $\phi_{\mathfrak{p}}$ as in §2. We are interested in computing $\beta_{\mathfrak{p}}(\phi_{\mathfrak{p}})$. Since $\Lambda_{\mathfrak{p}}$ is maximal we know $2\phi_{\mathfrak{p}} \in (O_{\mathfrak{p}})^n_n$, so by Lemmas 3.1 and 3.3 we have

$$\beta_{\mathfrak{p}}(\phi_{\mathfrak{p}}) = |\det(2\cdot 1_n)|_{\mathfrak{p}}^{\frac{n+1}{2}} \frac{e_{\mathfrak{p}}(2\phi_{\mathfrak{p}})}{2}. \tag{3.3}$$

By combining [Shi; Theorem 8.6(3), Proposition 3.9, (3.1.9)], we know the value of $\frac{1}{2}e_{\mathfrak{p}}(2\phi_{\mathfrak{p}})$. Therefore

$$\beta_{\mathfrak{p}}(\phi_{\mathfrak{p}}) = |\det(2\cdot 1_n)|_{\mathfrak{p}}^{\frac{n+1}{2}} q^{\kappa_{\mathfrak{p}} n} [\widetilde{\Lambda_{\mathfrak{p}}} : \Lambda_{\mathfrak{p}}]\xi, \tag{3.4}$$

where $q = \#(O_\mathfrak{p}/\mathfrak{p}O_\mathfrak{p})$, κ is defined by $2O_\mathfrak{p} = \mathfrak{p}_\mathfrak{p}^\kappa$,

$$\xi = \begin{cases} (1-q^{-m})\prod_{i=1}^{m-1}(1-q^{-2i}) & \text{if } t=0, \\ \prod_{i=1}^{m}(1-q^{-2i}) & \text{if } t=1, \\ (1+q^{-m})\prod_{i=1}^{m-1}(1-q^{-2i}) & \text{if } t=2, \mathfrak{p} \text{ is unramified in } K, \\ & \qquad \text{and } \widetilde{\Lambda_\mathfrak{p}} = \Lambda_\mathfrak{p}, \\ 2(1+q)(1+q^{1-m})^{-1}\prod_{i=1}^{m-1}(1-q^{-2i}) & \text{if } t=2, \mathfrak{p} \text{ is unramified in } K, \\ & \qquad \text{and } \widetilde{\Lambda_\mathfrak{p}} \neq \Lambda_\mathfrak{p}, \\ 2\prod_{i=1}^{m-1}(1-q^{-2i}) & \text{if } t=2, \text{and } \mathfrak{p} \text{ is ramified in } K, \\ 2(1+q)\prod_{i=1}^{m-1}(1-q^{-2i}) & \text{if } t=3, \\ 2(1+q)(1-q^{1-m})^{-1}\prod_{i=1}^{m-1}(1-q^{-2i}) & \text{if } t=4, \end{cases}$$

$m = \lfloor n/2 \rfloor$, $K = k(\sqrt{(-1)^{n/2}\det(\varphi)})$, and $\widetilde{\Lambda_\mathfrak{p}} = \{x \in V_\mathfrak{p} \mid 2\varphi_\mathfrak{p}(x, \Lambda_\mathfrak{p}) \in O_\mathfrak{p}\}$, For convenience, we also state [Shi, (3.1.9)] which says

$$[\widetilde{\Lambda_\mathfrak{p}} : \Lambda_\mathfrak{p}] = |\det(2\phi_\mathfrak{p})|_\mathfrak{p}^{-1}, \tag{3.5}$$

for a maximal lattice $\Lambda_\mathfrak{p}$ and $\phi_\mathfrak{p}$ as in (2.9).

CHAPTER 4

ARCHIMEDIAN LOCAL FACTORS

In this section we calculate the archimedian local factors $\mathrm{vol}_C(C_v)$ appearing in the product formula (5.7) below. To do this, for each $v \in \mathbf{a}$ we write down a symmetric space $\mathfrak{Z}_v$ on which G_v acts transitively which is equipped with a non-zero G_v invariant volume form $\omega_{\mathfrak{Z}}$. We explicitly carry out the procedure in §2 using ω_G and $\omega_{\mathfrak{Z}}$ to construct a non-zero C_v invariant volume form ω_C on the fibre C_v of G_v over some chosen point $p_v \in \mathfrak{Z}_v$, and then evaluate $\int_{C_v} \omega_C$.

It will be important to know our G invariant volume form in some set of coordinates on G. For our calculations, we choose the coordinates given by the strictly lower triangular matrix entries. These are known to give coordinates on an open subset of G whose compliment has measure zero, and the associated coordinate 1-forms give a basis for the cotangent space. The matrix $g^{-1}dg$ is a G invariant matrix of 1-forms under left multiplication, and so the form

$$\gamma_n = \bigwedge_{i>k} (g^{-1}dg)_{ik} \tag{4.1}$$

gives a G invariant volume form on G. Since the space of such forms is 1 dimensional, any G invariant volume form will be a constant multiple of γ_n.

We now compute the induced form ω_G on G^{ϕ_v} defined in §2.

Calculation 4.1. *The induced form ω_G on $G^{\phi_v}_{\mathbb{R}}$ is given up to sign by*

$$\omega_G = \frac{1}{2^n}\gamma_n = \frac{1}{2^n}\prod_{l=1}^{n} \det_l(x)^{-1} \bigwedge_{i>k} dx_{ik}.$$

Proof. To compute ω_G, it suffices to compute any non-zero monomial Θ in $\mathcal{F}^*(\omega_T)$. To see this, choose a non-zero monomial $\Theta = f(x)\bigwedge_{(i,k)\in I} dx_{ik}$ for some indexing set I, and let $\omega = f(x)^{-1}\bigwedge_{(i,k)\notin I} dx_{ik}$ be its complimentary monomial. Then we see that $\mathcal{F}^*(\omega_T)\wedge\omega = \Theta\wedge\omega = \omega_X$ since ω has at least one differential dx_{ik} in common with each of the other terms in $\mathcal{F}^*(\omega_T)$, so (2.7) is satisfied.

We choose to calculate the monomial $\Theta = f(x)\bigwedge_{i\le k} dx_{ik}$. Since we are only interested in finding ω_G up to sign, it is enough to compute ω_G for $\phi_v = 1_n$.

From (2.5) we have $t = \mathcal{F}(x) = {}^t xx$ and so $\mathcal{F}^*(dt) = {}^t(dx)x + {}^t x(dx)$. Therefore

$$\begin{aligned}\mathcal{F}^*(\omega_T) &= \bigwedge_{i\le k}\left(\sum_j dx_{ji}x_{jk} + x_{ji}dx_{jk}\right)\\ &= \Theta + \text{other terms}.\end{aligned} \tag{4.2}$$

We compute Θ by induction on the column bound k_0, showing that

$$\bigwedge_{i\le k\le k_0}\left(\sum_j dx_{ji}x_{jk} + x_{ji}dx_{jk}\right) = 2^{k_0}\bigwedge_{i\le k\le k_0}\sum_j x_{ji}dx_{jk} + \Psi \tag{4.3}$$

where Ψ is a sum of terms each of which has some dx_{ik} factor with $i > k$.

The case $k_0 = 1$ is obvious since the left side is just $2x_{11}dx_{11}$. If $k_0 > 1$ we have

(4.4)

$$\begin{aligned}&\bigwedge_{i\le k\le k_0}\left(\sum_j dx_{ji}x_{jk} + x_{ji}dx_{jk}\right)\\ &\quad= \bigwedge_{i\le k\le k_0-1}\left(\sum_j dx_{ji}x_{jk} + x_{ji}dx_{jk}\right)\wedge\bigwedge_{i\le k=k_0}\left(\sum_j dx_{ji}x_{jk_0} + x_{ji}dx_{jk_0}\right)\\ &\quad= \left(2^{k_0-1}\bigwedge_{i\le k\le k_0-1}\sum_j x_{ji}dx_{jk} + \Psi\right)\wedge\bigwedge_{i\le k=k_0}\left(\sum_j dx_{ji}x_{jk_0} + x_{ji}dx_{jk_0}\right)\end{aligned}$$

Now let us analyze the term $\Xi = \bigwedge_{i\le k_0}\left(\sum_j dx_{ji}x_{jk_0} + x_{ji}dx_{jk_0}\right)$ appearing at the end of (4.4). The only terms of Ξ contributing non-zero terms to Θ come from the column k_0. This is because all of the dx_{jk} terms with $k \leq k_0 - 1$ already appear in each term of $\bigwedge_{i \leq k \leq k_0 - 1} \sum_j x_{ji} dx_{jk}$ contributing to Θ, and so the wedge product of the two is zero. Also, since the entries of dx are linearly independent, such factors dx_{jk_0} must satisfy $j \leq k_0$ to contribute to Θ. So Ξ in (4.4) can be replaced by

$$\begin{aligned} \bigwedge_{i<k_0} \left(\sum_j x_{ji} dx_{jk_0} \right) \wedge \left(\sum_j dx_{jk_0} x_{jk_0} + x_{jk_0} dx_{jk_0} \right) \\ = 2 \bigwedge_{i \leq k_0} \left(\sum_j x_{ji} dx_{jk_0} \right). \end{aligned} \tag{4.5}$$

Doing this, we obtain (4.3) thus completing our proof. Our claim about Θ follows from (4.3) by taking $k_0 = n$. This together with Lemma A3 gives us

$$\begin{aligned} \Theta &= 2^n \bigwedge_{i \leq k} ({}^t x dx)_{ik} \\ &= 2^n \prod_{l=1}^{n} \det_l(x) \bigwedge_{i \leq k} dx_{ik} + \text{ other terms.} \end{aligned} \tag{4.6}$$

We choose $\omega = \omega_G$ as in (2.7) to be

$$\begin{aligned} \omega_G &= \frac{1}{2^n} \prod_{l=1}^{n} \det_l(x)^{-1} \bigwedge_{i>k} dx_{ik} \\ &\sim \frac{1}{2^n} \bigwedge_{i>k} ({}^t x dx)_{ik}, \end{aligned} \tag{4.7}$$

where $\sim$ denotes equivalence of forms restricted to G up to sign. We see that ω_G satisfies (2.7) since Lemma A2 gives

$$\omega_X = \bigwedge_{i,k} dx_{ik} \sim \det(x)^n \bigwedge_{i,k} dx_{ik} = \bigwedge_{i,k} ({}^t x dx)_{ik}. \quad \square \tag{4.8}$$

CHAPTER 4.1

LOCAL MASS FACTORS FOR $k_v = \mathbb{R}$ WITH φ DEFINITE

If v is real and φ_v is definite, then the change of basis in §2 gives $G_v^{\phi_v} = SO_n(\mathbb{R})$. Since $SO_n(\mathbb{R})$ is compact, $\tau_v(G_v)$ is finite. We now find the measure $\tau_{\mathbb{R}}$ of $SO_n(\mathbb{R})$ with respect to ω_G.

Letting $e_1 = (1, 0, \cdots, 0)$, there is a natural map $SO_n(\mathbb{R}) \longrightarrow \mathsf{S}^{n-1}$ sending $g \mapsto g(e_1)$. If we let $w_n = \bigwedge_{i=1}^n (g^{-1}dg)_{i1}$, we have $\gamma_n = w_n \wedge \gamma_{n-1}$. It is easy to check that w_n is the induced Riemannian volume form on S^{n-1} from $\mathsf{S}^{n-1} \hookrightarrow \mathbb{R}^n$ with the usual metric $\sum_i dx_i^2$ on $\mathbb{R}^n$. The volume of $\mathsf{S}^{n-1} \hookrightarrow \mathbb{R}^n$ is known to be:

$$\mathrm{vol}_{\mathbb{R}^n}(\mathsf{S}^{n-1}) = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}.$$

Let C be the fibre of this map over e_1, then γ_{n-1} gives the induced volume form on the fibre. For $n > 1$ this map is surjective with $C = \{1\} \times SO_{n-1}(\mathbb{R})$, but for $n = 1$ we have $SO_1(\mathbb{R}) = \{1\}$ which has $\frac{1}{2}$ the volume of the zero-sphere S^0.

This together with Calculation 4.1 gives

$$\begin{aligned} \tau_{\mathbb{R}}(G_{\mathbb{R}}) &= \frac{1}{2}\, 2^{-n} \prod_{l=1}^{n} \mathrm{vol}_{\mathbb{R}^l}(\mathsf{S}^{l-1}) \\ &= 2^{-(n+1)} \prod_{l=1}^{n} \frac{l\pi^{\frac{l}{2}}}{\Gamma(\frac{l}{2}+1)} \\ &= 2^{-(n+1)} \frac{n!\,\pi^{\frac{n(n+1)}{4}}}{\prod_{l=1}^{n}\Gamma(\frac{l}{2}+1)} \\ &= \frac{1}{2}\pi^{\frac{n(n+1)}{4}} \left(\prod_{l=1}^{n}\Gamma(l/2)\right)^{-1}. \end{aligned} \tag{4.1.1}$$

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CHAPTER 4.2

LOCAL MASS FACTORS FOR $k_v = \mathbb{R}$ WITH φ INDEFINITE

In this section we work with the normalized form ϕ_v of (2.9), and use q,r as defined there. We let $t = q - r$, and abbreviate $G_v^{\phi_v}$ as $G_{\mathbb{R}}$.

We define the (symmetric) space $\mathfrak{Z}_{\mathbb{R}}$ by

$$\mathfrak{Z}_{\mathbb{R}} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}_r^q \;\middle|\; x \in \mathbb{R}_r^r, y \in \mathbb{R}_r^t, {}^t x + x > {}^t y y \right\}.$$

To define a $G_{\mathbb{R}}$ action on $\mathfrak{Z}_{\mathbb{R}}$, let

$$B(z) = \begin{bmatrix} {}^t x & {}^t y & x \\ 0 & 1_t & y \\ -1_r & 0 & 1_r \end{bmatrix}, \qquad \gamma = \begin{bmatrix} \frac{-1}{\sqrt{2}}_r & 0 & \frac{1}{\sqrt{2}}_r \\ 0 & 1_t & 0 \\ \frac{1}{\sqrt{2}}_r & 0 & \frac{1}{\sqrt{2}}_r \end{bmatrix},$$

$$\mathfrak{Y} = \{ Y \in GL_n(\mathbb{R}) \mid {}^t Y \phi_v^{-1} Y = \operatorname{diag}[A, -B] \text{ with } A \in S_+^q, B \in S_+^r \},$$

and induce a $G_{\mathbb{R}}$ action on $\mathfrak{Z}_{\mathbb{R}}$ from the bijection

$$\begin{gathered} \mathfrak{Z}_{\mathbb{R}} \times GL_q(\mathbb{R}) \times GL_r(\mathbb{R}) \xrightarrow{\;\sim\;} \mathfrak{Y} \\ (z, \lambda, \mu) \longmapsto \gamma B(z) \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \end{gathered} \tag{4.2.1}$$

by allowing $\alpha \in G_{\mathbb{R}}$ to act on $\mathfrak{Y}$ by left multiplication. See [Shi2,§6] for details. Explicitly, (4.2.1) gives the action $z \mapsto \alpha z$ on $\mathfrak{Z}_{\mathbb{R}}$ by

$$\alpha \gamma B(z) = \gamma B(\alpha z) \begin{bmatrix} \lambda_\alpha(z) & 0 \\ 0 & \mu_\alpha(z) \end{bmatrix}, \tag{4.2.2}$$

for some matrices $\lambda_\alpha(z), \mu_\alpha(z)$.

Choosing a distinguished point $p_{\mathbb{R}} = \begin{bmatrix} 1_r \\ 0_r^t \end{bmatrix} \in \mathfrak{Z}_{\mathbb{R}}$ defines a map $F_{\mathbb{R}}$ by

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$$G_{\mathbb{R}} \xrightarrow{F_{\mathbb{R}}} \mathfrak{Z}_{\mathbb{R}} \tag{4.2.3}$$
$$\alpha \longmapsto \alpha p_{\mathbb{R}}.$$

If we write $\alpha \in G_{\mathbb{R}}$ as

$$\alpha = \begin{bmatrix} a & b & c \\ g & e & f \\ h & l & d \end{bmatrix} \tag{4.2.4}$$

with $a, d \in \mathbb{R}^r_r$ and $e \in \mathbb{R}^t_t$, then our map F sends

$$\alpha \longmapsto \alpha p_{\mathbb{R}} = \begin{bmatrix} (d-c)(d+c)^{-1} \\ (\sqrt{2})_t f(d+c)^{-1} \end{bmatrix}. \tag{4.2.5}$$

In these coordinates the stabilizer of $p_{\mathbb{R}}$ is given by

$$C_{\mathbb{R}} = \{\alpha \in G_{\mathbb{R}} \mid f = 0^r_t, c = 0^r_r\}. \tag{4.2.6}$$

For $\alpha \in C_{\mathbb{R}}$ the relation ${}^t x \phi_v x = \phi_v$ implies that l and h are also zero. Thus $C_{\mathbb{R}}$ decomposes as

$$C_{\mathbb{R}} \cong [G_{\mathbb{R}}(1_q) \times G_{\mathbb{R}}(1_r)] \cup \left[G^-_{\mathbb{R}}(1_q) \times G^-_{\mathbb{R}}(1_r)\right] \tag{4.2.7}$$
$$\alpha \mapsto \left(\begin{bmatrix} a & b \\ g & e \end{bmatrix}, d\right).$$

We choose the $G_{\mathbb{R}}$ invariant volume form on $\mathfrak{Z}_{\mathbb{R}}$ constructed in [Shi,§4.2], given by

$$\omega_{\mathfrak{Z}} = \delta(z)^{-n/2} \bigwedge_{i,k} dz_{ik} \tag{4.2.8}$$

where $\delta(z) = \det(\frac{1}{2}({}^t x + x - {}^t y y))$.

Computation of ω_C and $\int_C \omega_C$

We now compute the expression for ω_C on $C_{\mathbb{R}} = \mathrm{Stab}(p_{\mathbb{R}})$ described in §4. For this it is enough, by the last part of Lemma A6, for us to consider forms whose restrictions to the fibre $C_{\mathbb{R}}$ are equal up to sign. We write this equivalence as $\approx$.

From (4.2.5) we have

$$
\begin{aligned}
F_{\mathbb{R}}^*(dx) &= -(1_r + (d-c)(d+c)^{-1})dc(d+c)^{-1} \\
&\qquad\qquad + (1_r - (d-c)(d+c)^{-1})dd(d+c)^{-1} \\
&\approx -2_r\, dc\, d^{-1}, \\
F_{\mathbb{R}}^*(dy) &= -(\sqrt{2})_r df(d+c)^{-1} - (\sqrt{2})_r f(d+c)^{-1}d(d+c)(d+c)^{-1} \\
&\approx (\sqrt{2})_r df\, d^{-1}.
\end{aligned}
$$

Applying Lemma A2 and $\det(d) \approx 1$ to these gives

$$
\begin{aligned}
\bigwedge_{i,k} F_{\mathbb{R}}^*(dx)_{ik} &\approx 2^{r^2} \bigwedge_{i,k} dc_{ik}, \\
\bigwedge_{i,k} F_{\mathbb{R}}^*(dy)_{ik} &\approx 2^{\frac{rt}{2}} \bigwedge_{i,k} df_{ik},
\end{aligned}
$$

which together with the observation $\delta(p_{\mathbb{R}}) = 1$ yields

$$
F_{\mathfrak{Z}}^*(\omega_{\mathbb{R}}) \approx 2^{\frac{rn}{2}} \bigwedge_{i,k} dc_{ik} \bigwedge_{i,k} df_{ik}.
$$

We recall from Calculation 4.1,

$$
\omega_G \approx 2^{-n} \prod_{l=1}^{n} \det_l(\alpha)^{-1} \bigwedge_{i>k} d\alpha_{ik}.
$$

By the construction of ω_G in §2 and $F_{\mathbb{R}}^*(\omega_{\mathbb{R}})$ as above, and since the matrix $g^{-1}dg$ of §4 is skew symmetric. we see that the volume form ω_C on the fibre is

$$
\begin{aligned}
\omega_C &\approx 2^{\frac{-rn}{2}} 2^{-n} \prod_{l=1}^{n} \det_l(\alpha)^{-1} \bigwedge_{i>k} da_{ik} \bigwedge_{i>k} de_{ik} \bigwedge_{i,k} dg_{ik} \bigwedge_{i>k} dd_{ik} \\
&\approx 2^{\frac{-rn}{2}} \omega_{SO_q(\mathbb{R})} \wedge \omega_{SO_r(\mathbb{R})}.
\end{aligned}
$$

By comparison with ω_G in §4.1 and the isomorphism (4.2.7), we find that

$$
\begin{aligned}
\mathrm{vol}_C(C_{\mathbb{R}}) &= \int_{C_{\mathbb{R}}} |\omega_C| \\
&= 2 \cdot 2^{\frac{-rn}{2}} \left[\int_{SO_q(\mathbb{R})} \omega_{SO_q(\mathbb{R})} \right] \left[\int_{SO_r(\mathbb{R})} \omega_{SO_r(\mathbb{R})} \right] \\
&= 2 \cdot 2^{\frac{-rn}{2}} \frac{1}{2} \pi^{\frac{q(q+1)}{4}} \left(\prod_{k=1}^{q} \Gamma(k/2) \right)^{-1} \frac{1}{2} \pi^{\frac{r(r+1)}{4}} \left(\prod_{k=1}^{r} \Gamma(k/2) \right)^{-1},
\end{aligned}
$$

which completes our calculation.

CHAPTER 4.3

LOCAL MASS FACTORS FOR $k_v = \mathbb{C}$

In this section we work with the normalized form $\phi_v = 1_n$ of (2.9), and denote $G_v^{\phi_v}$ by $G_{\mathbb{C}}$. We define the (symmetric) space $\mathfrak{Z}_{\mathbb{C}}$ by

$$\mathfrak{Z}_{\mathbb{C}} = \{z \in \mathbb{R}_n^n \mid {}^t z = -z, {}^t zz < 1\}$$

and wish to define a $G_{\mathbb{C}}$ action on $\mathfrak{Z}_{\mathbb{C}}$. To do this we first define

$$B(z) = \begin{bmatrix} 1_n & z \\ -z & 1_n \end{bmatrix}, \qquad I = \begin{bmatrix} 1_n & 0 \\ 0 & -1_n \end{bmatrix},$$

$$\mathfrak{X} = \left\{ X \in GL_{2n}(\mathbb{R}) \;\middle|\; {}^t XIX = \begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix} \text{ with } A, B \in S_+^n \right\}.$$

We have an injection

$$\begin{aligned} \mathfrak{Z}_{\mathbb{C}} \times GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) &\longrightarrow \mathfrak{X} \\ (z, \lambda, \mu) &\longmapsto B(z) \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}. \end{aligned} \tag{4.3.1}$$

Writing $\alpha = a + bi \in G_{\mathbb{C}}$ with $a, b \in \mathbb{R}_n^n$, we define $\iota(\alpha) = \left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$ and allow α to act on $x \in \mathfrak{X}$ by left multiplication by $\iota(\alpha)$

$$\alpha x = \iota(\alpha) x.$$

By a direct calculation we see that this gives a well-defined action on the image of (4.3.1) and can be used to define a $G_{\mathbb{C}}$ action on $\mathfrak{Z}_{\mathbb{C}}$ by

$$\alpha B(z) = \iota(\alpha) B(z) = B(\alpha z) \begin{bmatrix} \lambda_\alpha(z) & 0 \\ 0 & \mu_\alpha(z) \end{bmatrix}, \tag{4.3.2}$$

the key observation being that ${}^t\iota(\alpha)I\iota(\alpha) = I$ for $\alpha \in G_{\mathbb{C}}$. The same calculation shows that

$$\lambda_\alpha(z) = \mu_\alpha(z) = (a + bz),$$

which we henceforth denote by $\mu_\alpha(z)$.

We choose a distinguished point $p_{\mathbb{C}} = 0_n^n \in \mathfrak{Z}_{\mathbb{C}}$. This defines a map

$$\begin{aligned} G_{\mathbb{C}} &\xrightarrow{F_{\mathbb{C}}} \mathfrak{Z}_{\mathbb{C}} \\ \alpha &\longmapsto \alpha p_{\mathbb{C}}. \end{aligned} \tag{4.3.3}$$

Writing this map out in real coordinates we see

$$\alpha = a + bi \longmapsto -ba^{-1}, \tag{4.3.4}$$

where $a, b \in \mathbb{R}_n^n$. In these coordinates the stabilizer of $p_{\mathbb{C}}$ is given by

$$C_{\mathbb{C}} = \mathrm{Stab}(p_{\mathbb{C}}) = \{\alpha = a + bi \in G_{\mathbb{C}} \mid b = 0_n^n\} \cong SO_n(\mathbb{R}). \tag{4.3.5}$$

We now construct a $G_{\mathbb{C}}$ invariant volume form on $\mathfrak{Z}_{\mathbb{C}}$. To do this we need to know how the differentials transform under the map $F_{\mathbb{C}}$. We begin with a few definitions. For any two points $w, z \in \mathfrak{Z}_{\mathbb{C}}$ we let

$$\xi(w, z) = 1_n - {}^t w z, \qquad \xi(z) = \xi(z, z), \tag{4.3.6}$$

$$\delta(w, z) = \det(\xi(w, z)), \qquad \delta(z) = \delta(z, z). \tag{4.3.7}$$

Then we have the relations

$${}^t B(w) I B(z) = \begin{bmatrix} \xi(w, z) & z + {}^t w \\ z + {}^t w & -\xi(w, z) \end{bmatrix} \tag{4.3.8}$$

From (4.3.8), ${}^t\iota(\alpha)I\iota(\alpha) = I$, and (4.3.2), we have

$${}^t\mu_\alpha(w)(\alpha z - \alpha w)\mu_\alpha(z) = z - w,$$

$${}^t\mu_\alpha(w)\xi(\alpha w, \alpha z)\mu_\alpha(z) = \xi(w, z).$$

Fixing $w \in \mathfrak{Z}_{\mathbb{C}}$, we differentiate these with respect to z and evaluate at $z = w$ to obtain

$$d(\alpha z) = {}^t\mu_\alpha(z)^{-1}\, dz\, \mu_\alpha(z)^{-1},$$

$$\delta(\alpha z) = \det(\mu_\alpha(z))^{-2}\delta(z).$$

By combining these two equations and using Lemma A4, we see that the expression

$$\omega_{\mathfrak{Z}} = \delta(z)^{\frac{1-n}{2}} \bigwedge_{i>k} dz_{ik} \tag{4.3.9}$$

is a non-zero $G_{\mathbb{C}}$ invariant volume form on $\mathfrak{Z}_{\mathbb{C}}$.

Computation of ω_C and $\int_C \omega_C$

We now compute the form ω_C on $C_{\mathbb{C}} = \mathrm{Stab}(p_{\mathbb{C}})$ described in §4. By the last part of Lemma A6, it is enough to consider forms whose restrictions to the fibre $C_{\mathbb{C}}$ are equal up to sign. We write this equivalence as $\approx$.

First we compute $F_{\mathbb{C}}^*(\omega_{\mathfrak{Z}})$. From (4.3.4) we have

$$\begin{aligned} F_{\mathbb{C}}^*(dz) &= -db\, a^{-1} - b\, d(a^{-1}) \\ &\approx db\, a^{-1}, \end{aligned}$$

and so

$$\bigwedge_{i>k} F_{\mathbb{C}}^*(dz)_{ik} \approx \bigwedge_{i>k} \left(db\, a^{-1}\right)_{ik}.$$

From the relations defining $G_{\mathbb{C}}$, we know that ${}^ta \approx a^{-1}$ and the restriction of ${}^ta\, db$ to $C_{\mathbb{C}}$ is skew symmetric, therefore so is $a({}^ta\, db)a^{-1} = db\, a^{-1}$. Applying Lemma A5 to this gives

$$\bigwedge_{i>k} db_{ik} = \prod_{l=1}^{n-1} \det{}_l(a) \bigwedge_{i>k} (db\, a^{-1})_{ik}$$

and so

$$F_{\mathbb{C}}^*(\omega_{\mathfrak{Z}}) = \prod_{l=1}^{n-1} \det{}_l(a)^{-1} \bigwedge_{i>k} db_{ik}$$

since $\delta(p_{\mathbb{C}}) = 1$.

From our choice of local measure in §2, the real volume form $\widetilde{\omega}$ associated to the complex volume form ω is given by $\omega \wedge \overline{\omega}$. Combining this with Calculation 4.1 we have

$$\begin{aligned}\widetilde{\omega_G} &= 2^{-2n} \prod_{l=1}^{n} \det_l(z)^{-1} \det_l(\bar{z})^{-1} \bigwedge_{i>k} (idz_{ik} \wedge d\bar{z}_{ik}) \\ &= 2^{\frac{n(n-5)}{2}} \prod_{l=1}^{n} \det_l(z)^{-1} \det_l(\bar{z})^{-1} \bigwedge_{i>k} (da_{ik} \wedge db_{ik}) \\ &\approx 2^{\frac{n(n-5)}{2}} \prod_{l=1}^{n-1} \det_l(a)^{-2} \bigwedge_{i>k} (da_{ik} \wedge db_{ik}) \,.\end{aligned}$$

By the procedure in §2 for $\widetilde{\omega_G}$ and $F_{\mathbb{C}}^*(\omega_3)$ as above, we see the (real) volume form ω_C on the fibre is given by

$$\omega_C = 2^{\frac{n(n-5)}{2}} \prod_{l=1}^{n-1} \det_l(a)^{-1} \bigwedge_{i>k} da_{ik}.$$

From §4.1, we know

$$\int_{SO_n(\mathbb{R})} \omega_G = \frac{1}{2}\pi^{\frac{n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)^{-1},$$

so we have

$$\mathrm{vol}_C(C_{\mathbb{C}}) = \int_{C_{\mathbb{C}}} \omega_C = 2^{\frac{n(n-3)}{2}} \int_{SO_n(\mathbb{R})} \omega_G = 2^{\frac{n(n-3)}{2}} \left(\frac{1}{2}\pi^{\frac{n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)^{-1} \right),$$

which completes our calculation.

CHAPTER 5

THE MASS FORMULA

In this section we compute an exact mass formula for the genus of a maximal lattice $\Lambda \subset (V,\varphi)$. We call a lattice $\Lambda \subset (V,\varphi)$ a **maximal lattice** if $\varphi(\Lambda) \subseteq O_k$ and Λ is maximal with this property.

In order to define the mass of the genus of Λ, we first define symmetric spaces $\mathfrak{Z}_v$ for all $v \in \mathbf{a}$. If v is real and φ_v is definite, then we define $\mathfrak{Z}_v$ to be a single point with measure one. If v is real and φ_v is indefinite or v is complex, then we define $\mathfrak{Z}_v$ as in §4.2 or §4.3 respectively. The spaces $\mathfrak{Z}_v$ come equipped with a transitive G_v action and a distinguished point p_v. We use this to define a surjective map

$$(5.1) \qquad \begin{aligned} F_v : G_v &\longrightarrow \mathfrak{Z}_v \\ \alpha &\longmapsto \alpha p_v \end{aligned}$$

and denote by C_v the fibre of F_v over p_v. We let

$$(5.2) \qquad \mathfrak{Z} = \prod_{v\in\mathbf{a}} \mathfrak{Z}_v, \qquad C = \prod_{v\in\mathbf{a}} C_v, \qquad p = (p_v)_{v\in\mathbf{a}},$$

and let F denote the product map

$$(5.3) \qquad F : G_{\mathbf{a}} \longrightarrow \mathfrak{Z}.$$

We observe that the $C = F^{-1}(p)$ is the fibre of F over p.

We define the **mass** of a quadratic form (V,φ) with respect to a lattice Λ to be

$$(5.4) \qquad \operatorname{Mass}(\Lambda,\varphi) = \sum_{a\in\mathfrak{Cl}} \nu(\Gamma^a)$$

where

$$\nu(\Gamma^a) = \begin{cases} [\Gamma^a : 1]^{-1} & \text{if } G_{\mathbf{a}} \text{ is compact,} \\ [\Gamma^a \cap \{\pm 1\} : 1]^{-1} \mathrm{vol}(\Gamma^a \backslash \mathfrak{Z}) & \text{otherwise.} \end{cases} \tag{5.5}$$

We now compute Mass(Λ, φ) in the case where the lattice Λ is maximal. By Lemma A7 applied to F, for each class a in the genus of Λ we have

$$\tau_{\mathbf{a}}(\Gamma^a \backslash G_{\mathbf{a}}) = \mathrm{vol}_C((\Gamma^a \cap S)\backslash C_{\mathbf{a}}) \mathrm{vol}_{\mathfrak{Z}}(\Gamma^a \backslash \mathfrak{Z}),$$

where $S = \{g \in G_{\mathbf{a}} \mid gz = z \text{ for every } z \in \mathfrak{Z}\}$. By Lemma A1, $S = \{(\pm 1)_{v, v\in \mathbf{a}}\}$ so we have

$$\tau_{\mathbf{a}}(\Gamma^a \backslash G_{\mathbf{a}}) \mathrm{vol}_C(C_{\mathbf{a}})^{-1} = [\Gamma^a \cap \{\pm 1\} : 1]^{-1} \mathrm{vol}_{\mathfrak{Z}}(\Gamma^a \backslash \mathfrak{Z}). \tag{5.6}$$

This together with Theorem 2.1 and our previous calculations gives

$$\begin{aligned} \mathrm{Mass}(\Lambda, \varphi) &= 2|D_k|^{\frac{n(n-1)}{4}} \mathrm{vol}_C(C_{\mathbf{a}})^{-1} \prod_{v \in \mathbf{h}} \beta_v(\Lambda, \psi)^{-1} \\ &= 2|D_k|^{\frac{n(n-1)}{4}} \prod_{v \in \mathbf{a}} \mathrm{vol}_C(C_v)^{-1} \prod_{v\in \mathbf{h}} \beta_v(\Lambda, \psi)^{-1}. \end{aligned} \tag{5.7}$$

THEOREM 5.1. *Let (V, φ) be a non-degenerate quadratic space of dimension $n \geq 3$ defined over a number field k of degree d over $\mathbb{Q}$. Then the mass of (V, φ) with respect to a maximal lattice $\Lambda \subset (V, \varphi)$ is given by*

$$\mathit{Mass}(\Lambda, \varphi) = 2|D_k|^{\lfloor \frac{(n-1)^2}{4} \rfloor} \left[\prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} |D_k|^{\frac{1}{2}} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \zeta_k(2j) \right] [\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v | \mathfrak{e}} \lambda_v$$

$$\prod_{v \in \mathbf{a}} b_v^{\varphi} \prod_{v \text{complex}} \left(2^{-\frac{(n-1)(n-2)}{2}} \pi^{\frac{n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)^{-1} \right)$$

$$\begin{cases} 2^{-\left(\frac{n-1}{2}\right)d} & \text{if } 2 \nmid n, \\ |D_k|^{\frac{1}{2}} \left[(\frac{n}{2} - 1)! (2\pi)^{-\frac{n}{2}} \right]^d L(\frac{n}{2}, \chi) & \text{if } 2|n, \end{cases}$$

where r_v *and* t_v *are defined by the normalization of* φ_v *in §2,*

$$\Gamma_i(s) = \pi^{\frac{i(i-1)}{4}} \prod_{j=0}^{i-1} \Gamma(s - (j/2)),$$

$$\widetilde{\Lambda} = \{x \in V \mid 2\varphi(x, \Lambda) \in O_k\},$$

$$b_v^{\varphi} = 2^{\frac{r_v n}{2}} \pi^{\frac{(n-r_v)r_v}{2}} \Gamma_{r_v}(r_v/2)\Gamma_{r_v}(n/2)^{-1},$$

$\mathfrak{e}$ *is the product of all prime ideals for which* $\widetilde{\Lambda}_v \neq \Lambda_v$*,* $\zeta_k(s)$ *and* $L(s,\chi)$ *are zeta and L-functions over k,* χ *is the non-trivial Hecke character on Gal(K/k) associated to the extension K/k where* $K = k(\sqrt{(-1)^{n/2}\det(\varphi)})$*, and* λ_v *is defined by*

$$\lambda_v = \begin{cases} 1 & \text{if } t_v = 1, \\ 2^{-1}(1+q)^{-1}(1+q^{1-m})(1+q^{-m}) & \text{if } t_v = 2, \mathfrak{p} \text{ is unramified in } K, \\ & \quad \text{and } \widetilde{\Lambda_{\mathfrak{p}}} \neq \Lambda_{\mathfrak{p}}, \\ 2^{-1} & \text{if } t_v = 2, \text{ and } \mathfrak{p} \text{ is ramified in } K, \\ 2^{-1}(1+q)^{-1}(1-q^{-2m}) & \text{if } t_v = 3, \\ 2^{-1}(1+q)^{-1}(1-q^{1-m})(1-q^{-m}) & \text{if } t_v = 4, \end{cases}$$

where q is the norm of the prime ideal at $v \in \mathbf{h}$ *and* $m = \lfloor \frac{n}{2} \rfloor$*.*

Proof. To avoid excessive algebra, we prove this formula in 3 parts.

Part 1: First we prove the case where φ_v is a positive definite at all $v \in \mathbf{a}$. In this case $C_v = G_v$ for all $v \in \mathbf{a}$, so by (5.7) we have

$$\mathrm{Mass}(\Lambda, \varphi) = 2|D_k|^{\frac{n(n-1)}{4}} \prod_{v \in \mathbf{a}} \beta_v(\psi)^{-1} \prod_{v \in \mathbf{h}} \beta_v(\Lambda, \psi)^{-1}.$$

By (2.9), $\phi_v = {}^t\sigma_v \psi \sigma_v$ and $|\det(\sigma_v)|_v = \left(\frac{|\det(\phi_v)|_v}{|\det(\psi)|_v}\right)^{\frac{1}{2}}$. Combining this with Lemma 3.2 we have

$$2|D_k|^{\frac{n(n-1)}{4}} \prod_{v \in \mathbf{a}} \left(|\det(\psi)|_v^{\frac{-(n+1)}{2}} |\det(\phi_v)|_v^{\frac{n+1}{2}} \beta_v(\phi_v)^{-1} \right)$$
$$\prod_{v \in \mathbf{h}} \left(|\det(\psi)|_v^{\frac{-(n+1)}{2}} |\det(\phi_v)|_v^{\frac{n+1}{2}} \beta_v(\Lambda_v, \phi_v)^{-1} \right),$$

which by the product formula and $\det(\phi_v) = \pm 1$ for all $v \in \mathbf{a}$, gives

$$2|D_k|^{\frac{n(n-1)}{4}} \prod_{v\in\mathbf{a}} \beta_v(\phi_v)^{-1} \prod_{v\in\mathbf{h}} \left(|\det(\phi_v)|_v^{\frac{n+1}{2}} \beta_v(\Lambda_v, \phi_v)^{-1} \right).$$

Substituting (3.4) and (4.1.1), using (3.5), and noticing $\prod_{v|2} 2^{\kappa_v} = 2^n$, we get

$$2|D_k|^{\frac{n(n-1)}{4}} \left(2\pi^{\frac{-n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2) \right)^d [\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}}$$

$$\left(2^{-nd} \prod_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \zeta_k(2i) \prod_{v|\mathfrak{e}} \lambda_v \right) \begin{cases} 1 & \text{if } 2 \nmid n, \\ L(\frac{n}{2}, \chi) & \text{if } 2|n. \end{cases}$$

Rearranging terms, and using (3.5), we get

$$2|D_k|^{\frac{n(n-1)}{4}} \left(2^{-(n-1)d} \right) \left[\left(\pi^{\frac{-n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2) \right)^d \prod_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \zeta_k(2i) \right]$$

$$[\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v|\mathfrak{e}} \lambda_v \begin{cases} 1 & \text{if } 2 \nmid n, \\ L(\frac{n}{2}, \chi) & \text{if } 2|n, \end{cases}$$

$$= 2|D_k|^{\frac{n(n-1)}{4}} \left(2^{-(n-1)d} \right) \left[\prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \zeta_k(2j) \right]$$

$$[\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v|\mathfrak{e}} \lambda_v \begin{cases} 2^{\frac{n-1}{2}d} & \text{if } 2 \nmid n, \\ \left[2^{n-1}(\frac{n}{2}-1)!(2\pi)^{-\frac{n}{2}} \right]^d L(\frac{n}{2}, \chi) & \text{if } 2|n, \end{cases}$$

$$= 2|D_k|^{\frac{n(n-1)}{4}} \left[\prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \zeta_k(2j) \right] [\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v|\mathfrak{e}} \lambda_v$$

$$\begin{cases} 2^{-\left(\frac{n-1}{2}\right)d} & \text{if } 2 \nmid n, \\ \left[(\frac{n}{2}-1)!(2\pi)^{-\frac{n}{2}} \right]^d L(\frac{n}{2}, \chi) & \text{if } 2|n, \end{cases}$$

$$= 2|D_k|^{\lfloor \frac{(n-1)^2}{4} \rfloor} \left[\prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} D_k^{\frac{1}{2}} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \zeta_k(2j) \right] [\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v|\mathfrak{e}} \lambda_v$$

$$\begin{cases} 2^{-\left(\frac{n-1}{2}\right)d} & \text{if } 2 \nmid n, \\ D_k^{\frac{1}{2}} \left[(\frac{n}{2}-1)!(2\pi)^{-\frac{n}{2}} \right]^d L(\frac{n}{2}, \chi) & \text{if } 2|n. \end{cases}$$

Part 2: Now suppose that all $v \in \mathbf{a}$ are real, but perhaps φ_v is indefinite at some v. Take

$$b_v^\varphi = 2^{\frac{r_v n}{2}} \pi^{\frac{(n-r_v)r_v}{2}} \Gamma_{r_v}(r_v/2)\Gamma_{r_v}(n/2)^{-1}$$

as above where r_v is defined by the normalization of φ_v in (2.9). For each indefinite v, we add an additional factor of b_v^φ from the formula in part 1, which is seen by observing

$$\mathrm{vol}_C(C_v)^{-1} = \left(2\pi^{\frac{-n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)\right) b_v^\varphi$$

and that $b_v^\varphi = 1$ if v is definite. Combined with the previous formula this proves the case where all $v \in \mathbf{a}$ are real.

Part 3: Finally consider arbitrary $v \in \mathbf{a}$. We define $r_v = 0$ for v complex, and so for such v we have $b_v^\varphi = 1$. Since each complex place replaces two real places in the totally real formula, we again have a correction factor. The relevant calculation to check for v complex is

$$\mathrm{vol}_C(C_v)^{-1} = \left(2\pi^{\frac{-n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)\right)^2 \left(2^{-\frac{(n-1)(n-2)}{2}} \pi^{\frac{n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)^{-1}\right) b_v^\varphi.$$

This together with Part 2 proves the theorem. □

One interesting application of this is to the case of an indefinite quadratic form (Λ, φ) in $n \geq 3$ variables with Λ a maximal lattice. In this case our formula gives the volume of the quotient $\Gamma^a \backslash \mathfrak{Z}$ using known facts about the spinor classes and genus. The main fact we need is:

- *For* (Λ, φ) *indefinite with* $\dim(V) \geq 3$*, each spinor genus contains only one class. I.e., the classes and the spinor genera coincide.*

Corollary 5.2. *Let* (Λ, φ) *be an indefinite quadratic form with* $\dim(V) \geq 3$*,* D *the subgroup of* $G_{\mathbf{A}}$ *stabilizing* Λ*, and* Λ *a maximal lattice. Then*

$$vol(\Gamma^a \backslash \mathfrak{Z}) = \varepsilon\, [k_{\mathbf{A}}^\times : k^\times \sigma(D)]\, Mass(\Lambda, \varphi)$$

where σ *is the spinor norm map* $G^{\varphi}_{\mathbf{A}} \longrightarrow k^{\times}_{\mathbf{A}}/(k^{\times}_{\mathbf{A}})^2$ *(see [Shi, (2.1.1)]) and* ε *is either 1 or 2 depending on whether* $\dim(V)$ *is odd or even. If* k *has class number one, then*

$$vol(\Gamma^a\backslash\mathfrak{Z}) = \varepsilon\, Mass(\Lambda, \varphi).$$

PROOF. From the fact above and [Shi, Lemma 2.3(4)] we know that the number of classes is $[k^{\times}_{\mathbf{A}} : k^{\times}\sigma(D)]$. We also know that $\nu(\Gamma^a)$ is independent of the class a [Shi, Thrm 5.10(1)]. Finally, $-1 \in \Gamma^a$ exactly when $\det(-1_n) = 1$ which happens iff $2|\dim(V)$. This proves the first assertion.

For the second part, from [Shi, Lemma 2.5] we know that $k^{\times}_{\mathbf{A}}/k^{\times}\sigma(D)$ is a quotient of the ideal class group of k. Thus if the class number of k is one, then $[k^{\times}_{\mathbf{A}} : k^{\times}\sigma(D)] = 1$. $\square$

APPENDIX

It will be convenient to know a few lemmas about matrices of differentials. If we take $x = (x_{ij})$ to be a matrix of functions, then we define the matrix dx to be the matrix (dx_{ij}) of differentials of x.

Lemma A1. *Let $\mathfrak{Z}_v$ be a symmetric space of the type described in §4.2 or §4.3. Then $\{g \in G_v \mid gz = z$ for every $z \in \mathfrak{Z}_v\} = \{\pm 1_n\}$.*

Proof. This is the analagous statemant of [Shi2, Prop6.4(5)] for orthogonal groups, and has the same proof with obvious modifications. □

Lemma A2. *Let dx be an $r \times t$ matrix of linearly independent differentials and let dx' be related to dx by the matrix equation $dx' = a(dx)$ for some $r \times r$ constant matrix a. Then*

$$\bigwedge_{i,k} dx'_{ik} = \det(a)^t \bigwedge_{i,k} dx_{ik}.$$

Similarly, if $dx' = (dx)a'$ for some $t \times t$ constant matrix a', then

$$\bigwedge_{i,k} dx'_{ik} = \det(a')^r \bigwedge_{i,k} dx_{ik}.$$

Proof. This is well known, and follows from the action of a (a') on a column (row) vector. □

Lemma A3. *Let dx be an $n \times n$ matrix of linearly independent differentials and let dx' be related to dx by the matrix equation $dx' = a(dx)$ for some $n \times n$*

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constant matrix a. Then

$$\bigwedge_{i\le k} dx'_{ik} = \prod_{l=1}^{n} \det_l(a) \bigwedge_{i\le k} dx_{ik} + \sum \begin{pmatrix} \textit{terms containing at least one} \\ \textit{factor } dx_{ik} \textit{ with } i>k \end{pmatrix}.$$

PROOF. We write

$$\bigwedge_{i\le k} dx'_{ik} = \bigwedge_{k=1}^{n} \bigwedge_{i\le k} dx'_{ik}.$$

It will be enough to analyze the columns $k \ge k_0$, proving inductively that for each $1 \le k_0 \le n$ we have

$$\bigwedge_{\substack{i\le k\\ k\ge k_0}} dx'_{ik} = \prod_{l=k_0}^{n-1} \det_l(a) \bigwedge_{\substack{i\le k\\ k\ge k_0}} dx_{ik} + \Omega, \tag{A3.1}$$

where Ω is a sum of terms each containing at least one factor dx_{ik} with $i > k$.

If $k_0 = n$ then

$$\begin{aligned} \bigwedge_{i\le k_0} dx'_{ik_0} &= \bigwedge_{i\le k_0} \sum_j a_{ij} dx_{jk_0} \\ &= \bigwedge_{i\le k_0} \sum_{\sigma\in S_n} a_{i\sigma(i)} dx'_{\sigma(i)k_0} \\ &= \det(a) \bigwedge_{i\le k_0} dx_{ik_0} \end{aligned}$$

since the only non-zero terms in the wedge product come from permutations of the row index i.

Now proceeding inductively, we consider the row k_0 and assume (A3.1) for all $k > k_0$. Then

(A3.2)
$$\begin{aligned} \bigwedge_{\substack{i\le k\\ k\ge k_0}} dx'_{ik} &= \bigwedge_{i\le k_0} dx'_{ik_0} \wedge \bigwedge_{\substack{i\le k\\ k\ge k_0+1}} dx'_{ik} \\ &= \left(\bigwedge_{i\le k_0} \sum_j a_{ij} dx_{jk_0}\right) \wedge \left(\prod_{l=k_0+1}^{n-1} \det_l(a) \bigwedge_{\substack{i\le k\\ k\ge k_0+1}} dx_{ik} + \Omega\right). \end{aligned}$$

The terms dx_{jk_0} of $\bigwedge_{i\le k_0}\sum_j a_{ij}dx_{jk_0}$ with $j>k_0$ cannot contribute to the term $\bigwedge_{i\le k,k\ge k_0} dx_{ik}$ since the entries of dx are linearly independent. Therefore the only terms which contribute to it are the dx_{jk_0} with $j\le k_0$. These can be written as a sum over permutations on the row index i,

$$\begin{aligned}\bigwedge_{i\le k_0}\sum_{j\le k_0} a_{ij}dx_{jk_0} &= \bigwedge_{i\le k_0}\sum_{\sigma\in S_{k_0}} a_{i\sigma(i)}dx'_{\sigma(i)k_0}\\ &= \det_{k_0}(a)\bigwedge_{i\le k_0} dx_{ik_0}.\end{aligned}$$

Combining this with (A3.2), we prove (A3.1). Our lemma then follows from (A3.1) by taking $k_0=1$. □

LEMMA A4. *Let dx be a skew symmetric $n\times n$ matrices of differentials whose upper triangular coordinates are linearly independent. Suppose $dx'={}^t a(dx)a$ for some $n\times n$ constant matrix a. Then*

$$\bigwedge_{i>k} dx'_{ik}=\det(a)^{n-1}\bigwedge_{i>k} dx_{ik}.$$

PROOF. This is proved in the same way as Lemma 3.2, the only difference being that the computation for scalar matrices here gives $\det(a)^{n-1}$. □

LEMMA A5. *Let dx be a skew symmetric $n\times n$ matrices of differentials whose upper triangular coordinates are linearly independent. Suppose $dx'=(dx)a$ for some $n\times n$ constant matrix a. Then*

$$\bigwedge_{i>k} dx'_{ik}=\prod_{l=1}^{n-1}\det_l(a)\bigwedge_{i>k} dx_{ik}.$$

PROOF. Writing out the above in coordinates, we have $\bigwedge_{k<i} dx'_{ik}=\bigwedge_{k<i}\sum_j dx_{ij}a_{jk}$. We prove by induction that

(A5.1) $$\bigwedge_{\substack{k<i\\ i\ge i_0}} dx'_{ik}=\prod_{l=i_0}^{n-1}\det_l(a)\bigwedge_{\substack{k<i\\ i\ge i_0}} dx_{ik}$$

for all $1 \leq i_0 \leq n$.

In the case $i_0 = n$, the non-zero terms of $\bigwedge_{k<n} \sum_j dx_{nj} a_{jk}$ come from choosing one term $dx_{nj} a_{jk}$ for each k with no repetition among the j indices. Thus the j index is a permutation of the k index, and we have

$$\bigwedge_{k<n} \sum_{\sigma \in S_{n-1}} dx_{n\sigma(k)} a_{\sigma(k)k} = \det_{n-1}(a) \bigwedge_{k<n} \sum_j dx_{nk}.$$

Now suppose $i_0 < n$. By induction we have

$$\bigwedge_{\substack{k<i \\ i \geq i_0}} \sum_j dx_{ij} a_{jk} = \left(\bigwedge_{k<i_0} \sum_j dx_{i_0 j} a_{jk} \right) \wedge \left(\bigwedge_{\substack{k<i \\ i \geq i_0+1}} \sum_j dx_{ij} a_{jk} \right)$$

$$= \left(\bigwedge_{k<i_0} \sum_j dx_{i_0 j} a_{jk} \right) \wedge \left(\prod_{l=i_0+1}^{n-1} \det_l(a) \bigwedge_{\substack{k<i \\ i \geq i_0+1}} dx_{ik} \right).$$

By skew symetry of dx, we see that all of the terms in $\bigwedge_{k<i_0} \sum_j dx_{i_0 j} a_{jk}$ with $j \geq i_0$ would give zero when wedged together with $\bigwedge_{k<i, i \geq i_0+1} dx_{ik}$. Thus the only terms that contribute have the form

$$\sum_{\sigma \in S_{i_0}} dx_{i_0 \sigma(k)} a_{jk} = \det_{i_0-1}(a) \bigwedge_{k<i_0} dx_{i_0 k}$$

which together with the above proves (A5.1). Our result follows from (A5.1) by taking $i_0 = 1$. $\square$

We now state two basic lemmas about volume forms on manifolds.

LEMMA A6. *Let $F : X \to Y$ be a map of C^∞ manifolds of dimensions n and m respectively, with rank(F) $= m$. Suppose that X is a group acting on Y and the map F commutes with this action. Choose $p \in Y$ and let $C = F^{-1}(p)$ be the fibre over p. Given X invariant volume forms ω_X and ω_Y on X and Y respectively, we can define a unique volume form ω_C on C by choosing $\omega \in (\bigwedge^{n-m})^*(X)$ such that*

(A6.1) $$\omega \wedge F^*(\omega_Y) = \omega_X$$

and taking ω_C to be the restriction $\omega|_C$ of ω to C. Further, ω_C is C invariant and when computing ω_C it suffices to take forms on X with coefficients in the fibre C over p.

PROOF. In this situation, the forms on X are determined by their definition on any neighborhood, so it is sufficient to check locally on X.

Choose a point $q \in F^{-1}(p) \subset X$. Taking $y_1, \cdots, y_m$ to be a set of coordinates on Y in some neighborhood of p, we can pull these back to give coordinates $x_1, \cdots, x_m$ on some neighborhood of q in X. Since $F^{-1}(p)$ is a regular submanifold of X, we can extend these to give a complete set of coordinates $x_1, \cdots, x_n$ on a possibly smaller neighborhood of q. In these coordinates we have

$$\omega_X = f(x) \bigwedge_{i=1}^{n} dx_i, \tag{A6.2}$$

$$F^*(\omega_Y) = f_1(x) \bigwedge_{i=1}^{m} dx_i. \tag{A6.3}$$

From this we see that any ω on X satisfying (A6.1) must have the form

$$\omega = \frac{f(x)}{f_1(x)} \bigwedge_{i=m+1}^{n} dx_i + \sum \begin{pmatrix} \text{terms containing at least one} \\ \text{factor from } \{dx_1, \cdots, dx_m\} \end{pmatrix}. \tag{A6.4}$$

Such an ω exists and is a volume form since both ω_X and ω_Y are nowhere vanishing. Uniqueness of ω_C follows since $x_1, \cdots, x_m$ are constant on C, so all terms of (A6.4) except the first term vanish on C.

To see the C invariance of ω_C, let $c_0 \in C$ act on (A6.1). This gives

$$c_0^* \wedge F^*(\omega_Y) = \omega_X.$$

But by uniqueness of ω_C we have the second part of

$$c_0^*(\omega_C) = c_0^*(\omega)|_C = \omega_C,$$

so ω_C is C invariant.

The final assertion is easy, and can be checked in the coordinates $x_1, \cdots, x_n$ above. We write $f_1(x) = f_2(x) + f_2'(x)$ where $f_2'(x)$ has coefficients all of which are zero on C, and observe that the $f_2'(x)$ term disappears whether we restrict coefficients before or after choosing ω. □

Lemma A7. *Suppose we are in the setting of Lemma A6, and take some Fuchsian subgroup* $\Gamma \subseteq X$*. We let* μ_C, μ_X*, and* μ_Y *denote the measures associated to* ω_C, ω_X*, and* ω_Y *respectively. Then*

$$\mu_X(\Gamma\backslash X) = \mu_Y(\Gamma\backslash Y)\mu_C(\Gamma \cap S\backslash C),$$

where $S = \{x \in X \mid xy = y \text{ for every } y \in Y\}$*.*

Proof. This follows from our choice of measures on X,Y, and C. □

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